Global Identifiability of Differential Models

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Abstract

Many real-world processes and phenomena are modeled using systems of ordinary differential equations (ODEs). Such systems usually involve unknown parameters. Often one might want to know the values of some parameters due to their meaning or importance. Usually one tries to determine (identify) them by measuring some output data. However, due to the structure of a model, it might be impossible to determine them through measurements. Hence, for designing a model, it is crucial to know whether the parameters of interest in a given model are identifiable. The first natural step in solving this problem is to check whether these parameters can in principle be uniquely recovered from the output data (globally identified). It is a present challenge to develop a rigorously justified efficient algorithm that solves this problem. We solve this by first proving an equivalence of an analytic definition of global identifiability with an algebraic condition, which results in a deterministic algorithm to check global identifiability. To address the efficiency challenge, we then present and justify an algorithm that, given $0 < p < 1$, checks global identifiability with probability at least $p$.

1 Introduction

Many real-world processes and phenomena are often modeled using systems of parametric ordinary differential equations (ODEs). There are many challenges in designing such a model. In this paper, we address one of them: structural identifiability. Often one might want to know the values of some parameters due to their meaning or importance. Usually one tries to determine (identify) them by measuring some output data. However, due to the structure of a model, it might be impossible to determine them through measurements. Hence, for designing a model, it is crucial to know whether the parameters of interest in a given model are identifiable. The first natural step in solving this problem is to check whether these parameters can in principle be uniquely recovered from the output data (globally identified).

Whether the parameters of interest can in principle be recovered from the output data up to finitely many choices (locally identified) is a weaker condition. There has been remarkable progress resulting in efficient algorithms, including rigorously justified probabilistic algorithms, for checking local identifiability (see [11, 27, 14] and the references given there). If the parameters are not locally identifiable, then they are not globally identifiable. If they are locally identifiable, then it still remains to check whether they are globally identifiable. Thus, it is desirable, and remains a challenge, to have a

(A) rigorously justified and
(B) efficient
algorithm to check whether any given set of locally identifiable parameters is also globally identifiable.

Prior important developments in solving this problem include:

- Taylor series approaches [22] with termination bounds in several particular cases [30, 31, 19]. Such bounds lead to algorithms in these particular cases, and these algorithms can be practical if applied to systems of small sizes.

- Generating series approach based on composing the vector fields associated to the model equations as well as a recursive approach based on integrals [32]. See [33] for a comparison of the Taylor and generating series approaches. This method can be turned into a (practical) algorithm if (practical) termination criteria are developed, which is still an open problem.

- Differential algebra-based approaches. They can be divided in two groups. One possibility is to treat the parameters as functions with zero derivatives and use differential elimination (see, for example, [17]). This approach can be computationally feasible for systems of small sizes.

Another option is to treat the parameters as elements of the field of coefficients and produce so-called input-output equations. This approach is implemented in two software packages: DAISY [5, 23, 24] and COMBOS [20]. These algorithms produce correct answers on many inputs. An important problem to address in the future is an improvement of this algorithm that will produce correct answers on all inputs (see Example 3 for more details).

- A discussion about making the algorithm implemented in DAISY probabilistic was given in [5, Section 4]. A possibility for the future work is to estimate the probability of the correctness for such an algorithm.

To summarize, significant progress has been made in the existing works toward parts (A) and (B) of the main challenge. However, there has not yet been developed a general, reliable, and efficient algorithm.

The main contribution of the present article is to provide a general, reliable algorithm that is also efficient enough for moderate size problems. In particular, Theorem 1 connects an analytic definition of global identifiability with an algebraic criterion. We show that this results in a deterministic algorithm to check global identifiability, thus addressing part (A) of the challenge. Thus, Theorem 2 together with Algorithm 1 address part (B) of the challenge with a randomized algorithm: given any $0 < p < 1$, Algorithm 1 verifies global identifiability with probability $p$, which we rigorously justify.

Let us sketch our approach. Informally, identifiability problem can be formulated as a question about fibers of the map that sends the parameter values and the initial conditions of the system of ODEs to the output data, which are functions of the corresponding solution. This observation is formalized using differential algebra in Proposition 1. One way to analyze this map is to reduce it to a map between finite-dimensional spaces. We do this by replacing the output functions by truncations of their Taylor series. Theorem 1 provides a criterion to find the order of truncation that contains enough information for the identifiability checking. This criterion can be applied efficiently using rank computation due to Proposition 3. After that, the identifiability question is reduced to the question about the generic fiber of a map between finite-dimensional varieties. To significantly increase the efficiency at this step, instead of considering the generic fiber, we consider a fiber over a randomly chosen point. We estimate the probability of correctness of such an algorithm in Theorem 2. We do it by first carefully analyzing the set of special points of this map and then applying the Demillo-Lipton-Schwartz-Zippel lemma. After considering the fiber over a random point, the problem turns into checking the consistency of a system of polynomial equations and inequations. This can be performed using symbolic, symbolic-numeric, or numeric methods. It turns out that, in practice, Gr"obner bases computations are efficient enough for moderate-size problems that we encountered and significantly outperformed (unexpectedly!), for example, such numerical algebraic geometry software as Bertini [3] in several examples that we considered. Any new method of checking the consistency of a system of polynomial equations and inequations can be potentially used to make our algorithm even more efficient.
The paper is structured as follows. In Section 2, we give a precise statement of the global identifiability problem and illustrate it by several examples. In Section 3, we give an algebraic criterion for global identifiability. In Section 4, we give a probabilistic criterion for global identifiability. In Section 5, we give an algorithm based on the criteria developed in the previous two sections. In Section 6, we give several practical examples.

2 Identifiability Problem

In this section, we give a precise statement of the global identifiability problem of algebraic differential models. For this, we need several notions.

Definition 1 (Algebraic Differential Model). An algebraic differential model is a system

\[
\begin{aligned}
\Sigma &= \Sigma(x, u, y, \mu, x^*) : \\
x' &= f(x, \mu, u), \\
y &= g(x, \mu, u), \\
x(0) &= x^*,
\end{aligned}
\]

where \(f\) and \(g\) are rational functions over \(\mathbb{C}\).

The derivative with respect to time is denoted by ‘\(\cdot\)’. The \(n\)-vector \(x\) stands for the state variables. The scalar \(u\) stands for the input variable. The \(m\)-vector \(y\) stands for the output variables. The \(\lambda\)-vector \(\mu\) stands for the system parameters. Finally the \(n\)-vector \(x^*\) stands for initial value parameters.

Remark 1. We assume that there is only one input function \(u\), for the sake of a simpler presentation. However, all our results and proofs can be generalized straightforwardly to the case of several input functions. If the input \(u\) is fixed (known) or \(u\) just does not appear, then \(u\) can be simply omitted when our algorithms/theoretical results are used.

Notation 1. We will use the following notations frequently.

1. \(\theta = \mu \cup x^*\) and \(s = \lambda + n\).
2. Let \(C^\infty(0)\) denote the set of all functions that are complex analytic in some neighborhood of \(t = 0\).
3. Let \(\Omega\) denote the set of all \((\hat{\theta}, \hat{u}) \in \mathbb{C}^s \times C^\infty(0)\) such that none of the denominators of \(f\) and \(g\) in \(\Sigma\), after the substitution of \((\hat{\theta}, \hat{u})\) into \((\theta, u)\) vanishes at \(t = 0\).
4. It is well known [12, Theorem 2.2.2] that for every \((\hat{\theta}, \hat{u}) \in \Omega\), there exists a unique solution over \(C^\infty(0)\) of the instance \(\Sigma(\theta, \hat{u})\). Let \(X(\hat{\theta}, \hat{u}), Y(\hat{\theta}, \hat{u})\) denote the unique solution.
5. For two sets \(A\) and \(B\), the notation \(A \subset B\) will be used to denote that \(A\) is a subset of \(B\) (not necessarily a proper subset). The notation \(A \subsetneq B\) will be used to denote that \(A\) is a subset of \(B\) and \(A \neq B\).

Definition 2 (Zariski Open). A subset \(U \subset C^\infty(0)\) is Zariski open if there is a differential polynomial \(P(z) \in \mathbb{C}\{z\}\) (see Section 3.1) such that \(U = \{f \in C^\infty(0) \mid P(f)|_{t=0} \neq 0\}\).

Remark 2. We call such a set Zariski open because \(C^\infty(0)\) can be thought of as a subset of the infinite-dimensional affine space of all formal power series and, in this space, the condition \(P(f)|_{t=0} \neq 0\) is an algebraic inequation.
**Definition 3** (Identifiability, see also [34, Section 2.6.1] and [1, Section II.B]). A parameter \( \theta \in \Theta \) is **globally (resp., locally) identifiable** if there exist Zariski open nonempty \( \Theta \subset \mathbb{C}^s \) and \( U \subset C^\infty(0) \) such that for every \( (\hat{\theta}, \hat{u}) \in (\Theta \times U) \cap \Omega \) the following set \( S(\hat{\theta}, \hat{u}) \) consists of one element (resp., finitely many elements):

\[
S_0(\hat{\theta}, \hat{u}) = \{ \hat{\theta} | \exists (\tilde{\theta}_2, \ldots, \tilde{\theta}_s) \text{ such that } (\hat{\theta}, \hat{u}) \in \Omega \text{ and } Y(\hat{\theta}, \hat{u}) = Y(\tilde{\theta}, \hat{u}) \}
\]

where we assumed that \( \theta = \theta_1 \) for notational simplicity.

**Remark 3.** The subset \( \theta^\# \subset \theta \) is said to be globally (resp., locally) identifiable if every parameter in \( \theta^\# \) is globally (resp., locally) identifiable.

Now we are ready to state the problem of identifiability.

**Problem 1** (Global Identifiability).

**In** \( \Sigma \) : an algebraic differential model given by rational functions \( f(x, \mu, u) \) and \( g(x, \mu, u) \)

**\( \theta^\ell \) : a subset of \( \theta = \mu \cup x^\ell \) of locally identifiable parameters

**Out** \( \theta^g \) : the set of all globally identifiable parameters in \( \theta^\ell \)

In the following we will illustrate Problem 1 on several simple examples.

**Example 1.** Consider the system

**In:**

\[
\Sigma := \begin{cases}
x_1' = \theta_1, \\
y_1 = x_1, \\
x_1(0) = \theta_2
\end{cases}
\]

**\( \theta^\ell := \{ \theta_1, \theta_2 \} \)**

**Out** \( \theta^g := \{ \theta_1, \theta_2 \} \)

Reason: We will explicitly show that \( \theta_1 \) is globally identifiable, thus also showing that it is locally identifiable (a proof that \( \theta_2 \) is globally identifiable can be done mutatis mutandis). For this, first note that \( Y(\theta) = \theta_1 t + \theta_2 \) and choose \( \Theta = \mathbb{C}^2 \). For all \( (\hat{\theta}_1, \hat{\theta}_2) \in \Theta \), we have

\[
S_{\theta_1}(\hat{\theta}_1, \hat{\theta}_2) = \{ \hat{\theta}_1 \in \mathbb{C} | \exists \tilde{\theta}_2 \in \mathbb{C} \text{ such that } \hat{\theta}_1 t + \tilde{\theta}_2 = \hat{\theta}_1 t + \hat{\theta}_2 \}
\]

\[
= \{ (\hat{\theta}_1 \in \mathbb{C} | \exists \tilde{\theta}_2 \in \mathbb{C} \text{ such that } \hat{\theta}_1 = \tilde{\theta}_1, \hat{\theta}_2 = \tilde{\theta}_2 \} = \{ \hat{\theta}_1 \}.
\]

Therefore, \( |S_{\theta_1}(\hat{\theta}_1, \hat{\theta})| = 1 \).

**Example 2.** Consider the system

**In:**

\[
\Sigma := \begin{cases}
x_1' = \theta_2^2, \\
y_1 = x_1, \\
x_1(0) = \theta_2
\end{cases}
\]

**\( \theta^\ell := \{ \theta_1, \theta_2 \} \)**
Example 3. Consider the system

\[
\begin{align*}
\Sigma &= \begin{cases} 
    x'_1 = 0, \\
    y_1 = x_1, \\
    y_2 = \theta_1 x_1 + \theta_2^2, \\
    x(0) = \theta_2.
\end{cases} \\
\theta^g &= \{\theta_2\}
\end{align*}
\]

Out: \(\theta^g := \{\theta_2\}\)

Reason: To see that \(\theta_1\) is locally identifiable, note that \(Y(\theta) = \theta_1^2 t + \theta_2\) and choose \(\Theta = \mathbb{C}^2\). For all \((\hat{\theta}_1, \hat{\theta}_2) \in \Theta\),

\[
S_{\theta_1}(\hat{\theta}_1, \hat{\theta}_2) = \{\hat{\theta}_1 \in \mathbb{C} \mid \exists \hat{\theta}_2 \in \mathbb{C} \text{ such that } \hat{\theta}_1^2 t + \hat{\theta}_2 = \hat{\theta}_1^2 \}
\]

Therefore, \(|S_{\theta_1}(\hat{\theta}_1, \hat{\theta}_2)| \leq 2\). Similarly, we conclude that \(|S_{\theta_2}(\hat{\theta}_1, \hat{\theta}_2)| = 1\), thus showing that \(\theta^l = \{\theta_1, \theta_2\}\) and \(\theta_2 \in \theta^g\).

To see that \(\hat{\theta}_1 \notin \theta^g\), let \(\Theta \subset \mathbb{C}^2\) be non-empty open such that, for all \((\hat{\theta}_1, \hat{\theta}_2) \in \Theta\), \(|S_{\theta_1}(\hat{\theta}_1, \hat{\theta}_2)| = 1\). Since, for all \((\theta_1, \theta_2) \in \mathbb{C}^2\), \(\theta_1 \neq 0\) implies \(|S_{\theta_1}(\theta_1, \theta_2)| = 2\),

\[
\Theta \subset \{(\theta_1, \theta_2) \in \mathbb{C}^2 \mid \theta_1 = 0\},
\]

so \(\Theta\) cannot be a non-empty open set.

Reason to see that \(\theta_1\) is locally identifiable, note that

\[
Y(\theta, u) = \begin{pmatrix} \theta_2 \\ \theta_1 \theta_2 + \theta_1^2 \end{pmatrix}
\]

and \(\Omega = \mathbb{C}^2\) and choose \(\Theta = \mathbb{C}^2\). For all \((\hat{\theta}_1, \hat{\theta}_2) \in \Theta\), we have

\[
S_{\theta_1}(\hat{\theta}_1, \hat{\theta}_2) = \{\hat{\theta}_1 \in \mathbb{C} \mid \exists \hat{\theta}_2 \in \mathbb{C} \text{ such that } \hat{\theta}_1 \theta_2 + \hat{\theta}_2^2 = \hat{\theta}_1 \theta_2 + \hat{\theta}_2^2 \} = \{\hat{\theta}_1 \in \mathbb{C} \mid \hat{\theta}_1 \theta_2 + \hat{\theta}_2^2 = \hat{\theta}_1 \theta_2 + \hat{\theta}_2^2 \} = \{\hat{\theta}_1 \neq \hat{\theta}_1 - \hat{\theta}_2\},
\]

and so \(|S_{\theta_1}(\hat{\theta}_1, \hat{\theta}_2)| \leq 2\). Similarly, we conclude that \(|S_{\theta_2}(\hat{\theta}_1, \hat{\theta}_2)| = 1\), thus showing that \(\theta^l = \{\theta_1, \theta_2\}\) and \(\theta_2 \in \theta^g\).

To see that \(\theta_1\) is not globally identifiable, let \(\Theta \subset \mathbb{C}^2\) be non-empty open such that, for all \((\hat{\theta}_1, \hat{\theta}_2) \in \Theta\), \(|S_{\theta_1}(\hat{\theta}_1, \hat{\theta}_2)| = 1\). Formula (2) implies that, for all \((\theta_1, \theta_2) \in \mathbb{C}^2\),

\[
|S_{\theta_1}(\theta_1, \theta_2)| = 2 \iff \theta_1 = -\theta_1 - \theta_2.
\]

Therefore,

\[
\Theta \subset \{(\theta_1, \theta_2) \in \mathbb{C}^2 \mid \theta_1 = -\theta_1 - \theta_2\},
\]

so \(\Theta\) cannot be non-empty open.

However, the software tool DAISY, see [5, 24, 23], which is based on the approach of input-output equations, returned that it is \textit{globally identifiable}. 

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Remark 4. In the above elementary examples, we were able to solve the systems explicitly and decide the identifiability directly. We now give an example with a system to which explicit solutions are not known to us but we are still able to decide identifiability using algorithms based on this paper. Note that the flexibility of our approach in working with subsets of the parameters for the identification purposes is shown in this example.

Example 4. Consider the system (a predator-prey model)

\[
\begin{align*}
\Sigma &= \begin{cases} 
  x'_1 = \theta_1 x_1 - \theta_2 x_1 x_2, \\
  x'_2 = -\theta_3 x_2 + \theta_4 x_1 x_2, \\
  y_1 = x_1, \\
  x_1(0) = \theta_5, \\
  x_2(0) = \theta_6.
\end{cases}
\end{align*}
\]

\[\theta^e = \{\theta_1, \theta_3, \theta_4, \theta_5\}\]

Out: \[\theta^g := \{\theta_1, \theta_3, \theta_4, \theta_5\}\]

This has been determined by Algorithm 1. Additionally, a calculation based on differential elimination terminates a with representations for \(\theta_1, \theta_3, \theta_4\) that are consequences of \(\Sigma\) of the form

\[\theta_i = \frac{P_i(y, \ldots, y^{(5)})}{Q_i(y, \ldots, y^{(5)})}, \quad \text{for } i = 1, 3, 4.\]

Note that, if the values of \(\theta_1\) and \(\theta_3\) are known, we can also write in a simpler form:

\[\theta_4 = \frac{\theta_1 \theta_3 y^2 - \theta_3 y y' - y'' y + y'^2}{\theta_1 y^3 - y'^2}.\]

One can also show directly by definition that neither \(\theta_2\) nor \(\theta_6\) is locally identifiable.

3 Algebraic Criteria

In this section, the analytic definition of global identifiability from the previous section will be characterized algebraically. First we provide an equivalence in terms of field extensions in Proposition 1 of Section 3.3. This section culminates in Theorem 1 of Section 3.4 which gives a constructive algebraic criterion for global identifiability.

3.1 Basic Terminology

A derivation \(\delta\) on a commutative ring \(R\) is a map \(\delta : R \rightarrow R\) such that, for all \(a, b \in R\),

\[\delta(a + b) = \delta(a) + \delta(b), \quad \delta(ab) = \delta(a)b + a\delta(b).\]  (3)

For a domain \(R\), Quot\((R)\) denotes the field of fractions of \(R\). The derivation \(\delta\) can be extended uniquely to a derivation on Quot\((R)\) using the quotient rule. An ideal \(I \subset R\) is said to be a differential ideal if \(a' \in I\) for every \(a \in I\). A differential ideal generated by \(a_1, \ldots, a_n \in I\) is denoted by \([a_1, \ldots, a_n]\).

The ring of differential polynomials in \(z_1, \ldots, z_s\) with coefficients in \(\mathbb{C}\) is denoted by \(\mathbb{C}\{z_1, \ldots, z_s\}\). As a ring, it is the ring of polynomials in the algebraic indeterminates

\[z_1, \ldots, z_s, z'_1, \ldots, z'_s, z''_1, \ldots, z''_s, \ldots, z_1^{(q)}, \ldots, z_s^{(q)}, \ldots\]
A differential ring structure is defined by, for all \( i, \, 1 \leq i \leq s \) and \( q \geq 0 \),
\[
\left( z_i^{(q)} \right)' := z_i^{(q+1)}, \quad z_i^{(0)} := z_i
\]
and extended to \( \mathbb{C}\{z_1, \ldots , z_s\} \) by the Leibniz rule, additivity (see (3)) and \( c' = 0 \) for all \( c \in \mathbb{C} \). For example,
\[
(2z_1^2z_3 + 3z_5^6)' = 4z_1^2z_3z_3 + 2z_1^2z_3^6 + 3z_5^6.
\]
For all \( i, \, 1 \leq i \leq s, \) and \( P \in \mathbb{C}\{z_1, \ldots , z_s\} \), we define \( \text{ord}_{z_i} P \) to be the largest integer \( q \) such that \( z_i^{(q)} \) appears in \( P \) if such a \( q \) exists and \(-1\) if such a \( q \) does not exist. For non-empty \( S, T \subset R \), we define
\[
T : S^\infty = \{ r \in R \mid \text{there exist } s \in S \text{ and } n \in \mathbb{Z}\geq 0 \text{ such that } s^nr \in T \}.
\]
If \( T \) is an ideal of \( R \), then \( T : S^\infty \) is an ideal of \( R \). For subsets \( X \subset \mathbb{C}^n \) and \( J \subset \mathbb{C}[x_1, \ldots , x_n] \), we denote
\[
I(X) := \{ f \in \mathbb{C}[x_1, \ldots , x_n] \mid f(p) = 0 \text{ for all } p \in X \},
\]
\[
Z(J) := \{ p \in \mathbb{C}^n \mid f(p) = 0 \text{ for all } f \in J \}.
\]

### 3.2 Algebraic Preparation

We will start with introducing technical notation and proving several auxiliary statements in this subsection.

Fix \( \Sigma \) from Definition 1 (see Notation 1 as well). Let \( Q \) be the lcm of all denominators in \( f \) and \( g \). Then all these rational functions can be written as
\[
f_i = \frac{F_i}{Q} \text{ for } 1 \leq i \leq n, \text{ and } g_j = \frac{G_j}{Q} \text{ for } 1 \leq j \leq m.
\]
Denote \( F = (F_1, \ldots , F_n) \) and \( G = (G_1, \ldots , G_m) \). Let
\[
R := \mathbb{C}[\mu] \{ x, y, u \}
\]
considered as a differential ring with \( \mu' = 0 \). For all \( i, \, 1 \leq i \leq n \) and \( j, \, 1 \leq j \leq m \), we consider \( F_i, G_j, \) and \( Q \) as elements of \( \mathbb{C}[\mu,x,u] \subset R \). Let
\[
J = \{ r \in R \mid \text{for all } ((\hat{\mu}, \hat{x}^*), \hat{u}) \in \Omega \text{ we have } r(\hat{\mu}, X((\hat{\mu}, \hat{x}^*), \hat{u}), Y((\hat{\mu}, \hat{x}^*), \hat{u}), \hat{u}) = 0 \},
\]
which is a (differential) ideal in \( R \). For all \( ((\hat{\mu}, \hat{x}^*), \hat{u}) \in \Omega \), we have
\[
Q(\hat{\mu}, \hat{x}^*, \hat{u}) \neq 0.
\]
Therefore,
\[
Q \cdot P \in J \implies P \in J \tag{5}
\]
for every \( P \in R \).

**Lemma 1.** \( J \cap \mathbb{C}[\mu, x]\{u\} = \{0\} \).

**Proof.** Assume that there is nonzero \( P(\mu, x, u', \ldots , u^{(h)}) \in J \cap \mathbb{C}[\mu, x]\{u\} \). Then there exist \( \hat{\theta} := (\hat{\mu}, \hat{x}^*) \in \mathbb{C}^s \) and \( \hat{u} \in \mathbb{C}^\infty(0) \) such that
\[
Q(\hat{\mu}, \hat{x}^*, \hat{u}(0)) \neq 0 \text{ and } P(\hat{\mu}, \hat{x}^*, \hat{u}(0), \hat{u}'(0), \ldots , \hat{u}^{(h)}(0)) \neq 0,
\]
which contradicts (4). \( \square \)
Lemma 2. We have
\[ J = \left( (Q x_i^j - F_i)^{(j)} , (Q y_k - G_k)^{(j)} \right) \mid 1 \leq i \leq n, 1 \leq k \leq m, j \geq 0 \] : Q^\infty
(6)
and J is a prime ideal.

Proof. Consider any ordering > of the variables in R such that
(1) \( y_{i}^{(k+1)} > y_{j}^{(k)} \) for every \( 1 \leq i, j \leq m \) and \( k \geq 0 \);
(2) \( y_{i}^{(k)} \) is larger that any variable in \( \mathbb{C}[\mu, \{x, u\}] \) for every \( 1 \leq i \leq m \) and \( k \geq 0 \);
(3) \( x_{i}^{(k+1)} > x_{j}^{(k)} \) for every \( 1 \leq i, j \leq n \) and \( k \geq 0 \);
(4) \( x_{i}^{(k)} \) is larger that any variable in \( \mathbb{C}[\mu, \{u\}] \) for every \( 1 \leq i \leq n \) and \( k \geq 0 \);

Then the set of polynomials
\[ S := \left\{ (Q x_i^j - F_i)^{(j)} , (Q y_k - G_k)^{(j)} \mid 1 \leq i \leq n, 1 \leq k \leq m, j \geq 0 \right\} \]
(7)
is a triangular set (see [13, Definition 4.1 and page 10]).

In order to prove (6), we consider \( \tilde{J} := (S) : Q^\infty \). For all \( ((\hat{\mu}, \hat{x}^*), \hat{u}) \in \Omega \), it follows from the definition of
\[ X((\hat{\mu}, \hat{x}^*), \hat{u}), Y((\hat{\mu}, \hat{x}^*), \hat{u}) \]
that
\[ Q(\hat{\mu}, \hat{x}^*, \hat{u}) \cdot X((\hat{\mu}, \hat{x}^*), \hat{u})' - F(\hat{\mu}, X((\hat{\mu}, \hat{x}^*), \hat{u}), \hat{u}) = 0, \]
\[ Q(\hat{\mu}, \hat{x}^*, \hat{u}) \cdot Y((\hat{\mu}, \hat{x}^*), \hat{u}) - G(\hat{\mu}, X((\hat{\mu}, \hat{x}^*), \hat{u}), \hat{u}) = 0. \]

Since J is a differential ideal, we therefore have \( S \subset J \). By (5), we moreover obtain \( \tilde{J} \subset J \). Consider \( P \in J \). Let \( N \) be a positive integer and \( P_0 \in \mathbb{C}[\mu, x] \{u\} \) (see [13, Section 4.2]) such that \( Q^N P - P_0 \in \tilde{J} \).
Hence, \( P_0 \in J \), so \( P_0 = 0 \) by Lemma 1. Then \( Q^N P \in \tilde{J} \), so \( P \in \tilde{J} \).

For the primality of \( J \), consider \( P_1 \) and \( P_2 \) such that \( P_1 \cdot P_2 \in J \). Let \( N \) be such that \( Q^N P_1 \) and \( Q^N P_2 \) are equivalent to elements \( \tilde{P}_1 \) and \( \tilde{P}_2 \) of \( \mathbb{C}[\mu, x] \{u\} \) modulo \( J \). If \( \tilde{P}_1 \tilde{P}_2 = 0 \), then \( P_1 \in J \) or \( P_2 \in J \). If \( \tilde{P}_1 \tilde{P}_2 \neq 0 \), then, by Lemma 1, \( Q^N P_1 \cdot Q^N P_2 \notin J \), so \( P_1 P_2 \notin J \).

Lemma 3. Let \( P(\theta, u, \ldots, u^{(N)}) \in \mathbb{C}[\theta] \{u\} \) be nonzero. Then there exist nonempty Zariski open subsets \( \Theta \in \mathbb{C}^s \) and \( U \subset C^\infty(0) \) such that, for every \( \theta^* \in \Theta \) and \( u^* \in U \), the function \( P(\theta^*, u^*, \ldots, (u^*)^{(N)}) \) is a nonzero element of \( C^\infty(0) \).

Proof. We write \( P(\theta, u, \ldots, u^{(N)}) = \sum_{i=1}^{\ell} c_i(\theta) m_i(u) \), where \( m_1, \ldots, m_\ell \) are distinct monomials from \( \mathbb{C}\{u\} \) and \( c_1(\theta), \ldots, c_\ell(\theta) \in \mathbb{C}[\theta] \). Let \( W(u) \in \mathbb{C}\{u\} \) be the determinant of the Wronskian matrix of \( m_1, \ldots, m_\ell \). We set
\[ \Theta := \{ \theta \in \mathbb{C}^s \mid c_1(\theta) \neq 0 \} \]
and \( U = \{ u \in C^\infty(0) \mid W(u)_{|t=0} \neq 0 \} \).

Since \( c_1(\theta) \) is a nonzero polynomial, \( \Theta \neq \emptyset \). The differential polynomial \( W \) can be considered as an algebraic polynomial \( W(u, u', \ldots, u^{(M)}) \in \mathbb{C}[u, u', \ldots, u^{(M)}] \) for some \( M \). Let \( (a_0, \ldots, a_M) \in \mathbb{C}^{M+1} \) be such that \( W(a_0, \ldots, a_M) \neq 0 \). We set \( u^*(t) := \sum_{i=0}^{M} a_i t^i \). A direct computation shows that
\[ W(u^*, \ldots, (u^*)^{(M)})_{|t=0} = W(a_0, \ldots, a_M) \neq 0. \]

Thus, \( u^* \in U \), so \( U \neq \emptyset \). Let \( \theta^* \in \Theta \) and \( u^* \in U \). If the function \( P(\theta^*, u^*, \ldots, (u^*)^{(N)}) \) zero, it provides a nontrivial (because of \( c_1(\theta^*) \neq 0 \)) linear dependence of \( m_1(u^*), \ldots, m_\ell(u^*) \). Since \( W(u^*) \neq 0 \), such a dependence does not exist due to [18, Proposition 2.8].

\[ \square \]
Lemma 4. Let \( \varphi : R \to \mathbb{C} \) be a \( \mathbb{C} \)-algebra homomorphism such that \( \varphi(Q) \neq 0, J \subset \text{Ker} \varphi \), and there exists \( C > 0 \) such that \( |\varphi(u^{(i)})| < C \) for all \( i \geq 0 \). We define power series

\[
u_{\varphi}(t) = \sum_{i=0}^{\infty} \frac{\varphi(u^{(i)})}{i!} t^i, \quad y_{\varphi} = \sum_{i=0}^{\infty} \frac{\varphi(y^{(i)})}{i!} t^i.
\]

Then these series define functions in \( C^{\infty}(0) \) and

\[
y_{\varphi} = Y(\varphi(\theta), u_{\varphi}).
\]

Proof. Since, for all \( i \geq 0, |\varphi(u^{(k)})| < C \), the power series \( u_{\varphi} \) converges for all \( t \in \mathbb{C} \) and defines an element of \( C^{\infty}(0) \). Moreover, a direct computation shows that \( u_{\varphi}^{(k)}(0) = \varphi(u^{(k)}) \) for every \( k \geq 0 \). We also define \( x_{\varphi} := \sum_{i=0}^{\infty} \frac{\varphi(x^{(i)})}{i!} t^i \).

By the theorem of existence and uniqueness of solutions for differential equations [12, Theorem 2.2.2], there exist unique \( X((\varphi(\mu), \varphi(x)), u_{\varphi}), Y((\varphi(\mu), \varphi(x)), u_{\varphi}) \in C^{\infty}(0) \) satisfying the instance \( \Sigma((\varphi(\mu), \varphi(x)), u_{\varphi}) \), as in Notation 1. We denote these functions by \( \hat{x} \) and \( \hat{y} \), respectively.

We prove that \( (\hat{x}_{\varphi})^{(j)}(0) = \varphi(x^{(j)}) \) for every \( 1 \leq i \leq n \) and \( j \geq 0 \) by induction on \( j \). The base case is \( j = 0 \). Then \( (\hat{x}_{\varphi})^{(0)}(0) = \varphi(x^{(0)}) \), because \( \varphi(x) \) is the initial condition. Assume that \( (\hat{x}_{\varphi})^{(k)}(0) = \varphi(x^{(k)}) \) for every \( 1 \leq i \leq n \) and \( 0 \leq k \leq j \). We write the differential polynomial \( (Qx^{(j)}_{\mu} - F_{\mu})^{(j)} \) in the form \( Qx^{(j+1)}_{\mu} + P \), where \( P \) only involves derivatives of \( x \) of order at most \( j \). Since this differential polynomial belongs to \( J \),

\[
\varphi (x^{(j+1)}) = -\frac{P(x_{\mu})}{\varphi(Q)}.
\]

The inductive hypothesis implies that the right-hand side is equal to

\[
\frac{P(\hat{x}, \mu, u_{\varphi})}{Q(x, \mu, u_{\varphi})}|_{t=0} = (\hat{x})^{(j+1)}(0).
\]

Hence, \( \hat{x} \) and \( x_{\varphi} \) have the same Taylor expansion, so coincide. Using this, one can analogously prove that \( \hat{y} \) and \( y_{\varphi} \) coincide.

3.3 Algebraic Criterion: Non-constructive Version

Notation 2. Let \( S := R/J \) and \( F := \text{Quot}(S) \). Note that \( F \) is generated by the images of \( \mu, x \), and \( u, u', \ldots \). We denote the subfield of \( F \) generated by the image of \( \mathbb{C}\{y, u\} \) by \( E \).

In what follows, \( \theta \) will be understood as a tuple \( (\mu, x^{\ast}) \) if we talk about parameters of \( \Sigma \) and as a tuple of variables \( (\mu, x_1, \ldots, x_n) \) if we talk about \( R \) or its subalgebras.

Proposition 1. Parameter \( \theta \) in system \( \Sigma \) is locally (resp., globally) identifiable if and only if the field extension \( E \subset E(\theta) \) is algebraic (resp., these fields are equal).

Proof. First we prove the “if” part. Assume that \( E \subset E(\theta) \) is an algebraic extension (resp., these fields are equal). Then there exists a nonzero polynomial (resp., nonzero polynomial linear in \( \theta \)) in \( \mathbb{C}\{y, u\} \cap J \) such that its leading coefficient \( \ell \) does not lie in \( J \). There exists \( M \) such that \( Q^M \ell \) is equal to some \( \hat{\ell} \in \mathbb{C}\{\theta\} \{u\} \) modulo \( J \). We apply Lemma 3 to the polynomial \( P := Q\hat{\ell} \) and obtain nonempty open \( \Theta \subset \mathbb{C}^s \) and \( U \subset C^{\infty}(0) \). We claim that Definition 3 holds for this choice of open sets. Consider any \( (\theta^{\ast}, u^{\ast}) \in \Theta \times U \). The choice of \( \Theta \) and \( U \) implies that each of \( Q \) and \( \hat{\ell} \) do not vanish at \( (\theta^{\ast}, u^{\ast}) \). Then \( \ell \) does not vanish
at this point, so there are only finitely many (resp., only one) possible values for every \( \theta \) provided that \( y^* := Y(\theta^*, u^*) \) and \( u^* \) are fixed.

Now we prove the “only if” part for local identifiability. Assume the contrary: the parameter \( \theta \) is locally identifiable but \( \theta \in F \) is transcendental over \( E \). Let \( Q_1 \subseteq \mathbb{C}[\theta] \) and \( Q_2 \subseteq \mathbb{C}\{u\} \) be polynomials defining complements to \( \Theta \) and \( U \) from the definition of identifiability, respectively. Let

\[
S_0 := \mathbb{C}\{y, u\}/(J \cap \mathbb{C}\{y, u\}).
\]

Consider the finitely generated extension of rings \( S_0[\theta] \subset S_{QQ_1Q_2} \), where the latter is the localization of \( S \) at \( QQ_1Q_2 \). Let \( Q_3 \in S_0[\theta] \) be such that every homomorphism \( S_0[\theta]Q_3 \rightarrow \mathbb{C} \) can be extended to a homomorphism \( S_{QQ_1Q_2} \rightarrow \mathbb{C} \). Such a \( Q_3 \) exists by [29, Proposition 1.9.4]. If the former homomorphism is \( \mathbb{C} \)-linear, then the latter is also \( \mathbb{C} \)-linear. Consider any \( \mathbb{C} \)-algebra homomorphism \( \varphi: S_0 \rightarrow \mathbb{C} \) such that \( \varphi(Q_3) \) understood as the result of applying \( \varphi \) to the coefficients of \( Q_3 \) as a polynomial in \( \theta \) is a nonzero element of \( \mathbb{C}[\theta] \) and \( |\varphi(u^{(j)})| < 1 \) for all \( j \). Since \( \theta \) is transcendental over \( S_0 \) and \( \varphi(Q_3) \neq 0 \), this homomorphism can be lifted to infinitely many distinct homomorphisms \( \varphi_k: S_0[\theta_1]Q_3 \rightarrow \mathbb{C} \) and, consequently, to infinitely many distinct homomorphisms

\[
\varphi_k: S_{QQ_1Q_2} \rightarrow \mathbb{C}, \quad k = 1, 2, 3, \ldots
\]

Due to Lemma 4, for every \( k \), we have (see (8))

\[
y_{\varphi_k} = Y(\varphi_k(\theta), u_{\varphi_k}).
\]

Since the restrictions of all \( \varphi_k \) to \( S_0 \) coincide, \( y_{\varphi_k} \) and \( u_{\varphi_k} \) do not depend on \( k \). On the other hand, \( \varphi_k(\theta) \) are distinct.

Moreover, since we have localized at \( Q_1 \) and \( Q_2 \), \( \varphi_k(Q_1) \neq 0 \) and \( \varphi_k(Q_2) \neq 0 \). Therefore, \( \varphi_k(\theta) \in \Theta \) and \( u_{\varphi_k} \in U \) for all \( k \). Thus, the set \( S(\varphi_1(\theta), u_{\varphi_1}) \) from Definition 3 contains infinitely many distinct numbers \( \varphi_k(\theta) \), so the parameter \( \theta \) is not locally identifiable.

Now we prove “only if” part for global identifiability. Assume the contrary: the parameter \( \theta \) is globally identifiable, but \( \theta \in F \) does not lie in \( E \). Let \( Q, Q_1, Q_2, Q_3, \Theta, U \), and \( S_0 \) be the same as in the proof for local identifiability. Let \( P(z) \in S_0[z] \) be a minimal polynomial for \( \theta \) over \( S_0 \). Since \( \theta \notin E \), \( \deg P \geq 2 \). The element \( Q_3 \in S_0[\theta] \) can be written as \( P_0(\theta) \), where \( P_0(z) \in S_0[z] \) is of degree less than the degree of \( P(z) \). Let \( Q_3 \) denote the product of the resultant of \( P(z) \) and \( P_0(z) \) and the discriminant of \( P(z) \). Consider any homomorphism \( \varphi: S_0 \rightarrow \mathbb{C} \) such that \( \varphi(Q_4) \neq 0 \), and \( |\varphi(u^{(j)})| < 1 \) for all \( j \). Then \( \varphi(P) \) has at least two distinct roots that are not roots of \( \varphi(P_0) \). Hence, \( \varphi \) can be extended to two distinct homomorphisms \( \varphi_1, \varphi_2: S_0[\theta_1]Q_3 \rightarrow \mathbb{C} \). These homomorphisms can be further extended to \( \varphi_1, \varphi_2: S_{QQ_1Q_2} \rightarrow \mathbb{C} \). Due to Lemma 4, we have

\[
y_{\varphi_k} = Y(\varphi_k(\theta), u_{\varphi_k}), \quad k = 1, 2.
\]

Since we localized at \( Q_1 \) and \( Q_2 \), \( \varphi_k(\theta) \in \Theta \) and \( u_{\varphi_k} \in U \) for \( k = 1, 2 \). Since \( \varphi_1|S_0 = \varphi_2|S_0 \), we have \( y_{\varphi_1} = y_{\varphi_2} \) and \( u_{\varphi_1} = u_{\varphi_2} \). On the other hand, \( \varphi_1(\theta) \neq \varphi_2(\theta) \), so the parameter \( \theta \) is not globally identifiable.

3.4 Algebraic Criterion: Constructive Version

Although Proposition 1 provides us with an algebraic criterion, it involves the quotient field of an infinitely generated algebra, so is not constructive enough.

Notation 3. Let \( h = (h_1, \ldots, h_m) \in \mathbb{Z}^m_{\geq 0} \). We construct \( b(h) \in \mathbb{Z}^m_{\geq 0} \) and a set of differential polynomials \( S_h \) by the following procedure.
Proof. Lemma 5. For every tuple ordering described in the proof of Lemma 2 such that the free variables are exactly $x$.

Remark 5. For every $h$, $J_h$, being a subset of $S_h$, is a triangular set for $J_h$ with respect to every ordering described in the proof of Lemma 2 such that the free variables are exactly $x$, $\mu$, and $u_h$.

Lemma 5. For every tuple $h \in \mathbb{Z}^m_{\geq 0}$, $J \cap R_h = J_h$. In particular, $J_h$ is prime.

Proof. By Lemma 2, $J$ is prime. The statement now follows from [13, Proposition 4.5].

We denote the $i$-th standard basis vector in $\mathbb{Z}^m_{\geq 0}$ by $1_i$ and set

$$1 := 1_1 + \ldots + 1_m.$$  

For all $h \in \mathbb{Z}^m_{\geq 0}$, we denote the zero set of $J_h$ by $Z_h$.

Theorem 1. For all $h \in \mathbb{Z}^m_{\geq 0}$ such that

1. the projection of $Z_h$ to $(y_h, u_h)$-coordinates is dominant,

2. the projection of $Z_{h+1,i}$ to the $(y_{h+1,i}, u_{h+1,i})$-coordinates is not dominant for every $1 \leq i \leq m$,

for all non-empty subsets $\theta^\#$, we have

$$\left( \text{every parameter in } \theta^\# \text{ is globally identifiable} \right) \iff \left( \text{the generic fiber of the projection of } Z \text{ to } (y_{h+1,i}, u_{h+1}) \text{ is of cardinality one} \right),$$

where $Z$ is the Zariski closure of the projection of $Z_{h+1}$ to the subspace with coordinates $(\theta^\#, y_{h+1,i}, u_{h+1})$.

Remark 6. One can check efficiently whether $h$ satisfies requirements (1) and (2) from Theorem 1 using Proposition 3. Corollary 1 implies that such an $h$ always exists.

Lemma 6. Let $h$ be the tuple from the statement of Theorem 1. Then $E$, introduced in Notation 2, is generated by the image of $\mathbb{C}[y_{h+1}][u]$. 

Proof. Consider $i$, $1 \leq i \leq m$. Since the projection of $Z_{h+1,i}$ to the $(y_{h+1,i}, u_{h+1,i})$-coordinates is not dominant and the projection of $Z_h$ to the $(y_h, u_h)$-coordinates is dominant

$$J_{h+1,i} \cap \mathbb{C}[y_{h+1,i}, u_{h+1,i}] \neq \{0\} \text{ and } J_h \cap \mathbb{C}[y_h, u_h] = \{0\}.$$  

Consider a nonzero $P \in J_{h+1,i} \cap \mathbb{C}[y_{h+1,i}, u_{h+1,i}]$. Since $S_{h+1,i}$ is a triangular set, $J_{h+1,i} = (S_{h+1,i}) : Q^\infty$, and the ideal $J_{h+1,i}$ is prime, $S_{h+1,i}$ is a characteristic set of $J_{h+1,i}$ (see [13, Definitions 5.5 and 5.10,
The derivative of $Ay$. By the inductive hypothesis, the latter coincides with the subfield generated by the image of $y^{(h_i)}$. This implies that the image of $y^{(h_i)}$ in $F$ belongs to the subfield generated by the image of $C[y_{h_{k+1}}]\{u\}$, so the base case is proved. Let $k > 2$. Consider $i, 1 \leq i \leq m$. The inductive hypothesis implies that there are $A, B \in C[y_{h_{(k-1)}}]\{u\}$ such that $Ay^{(h_i+k-1)} + B = J$, but $A \not\in J$. Taking the derivative of $Ay^{(h_i+k-1)} + B$, we obtain

$$Ay^{(h_i+k)} + A'y^{(h_i+k-1)} + B' \in J.$$  

This implies that the image of $y^{(h_i+k)}$ in $F$ belongs to the subfield generated by the image of $C[y_{h_{k+1}}]\{u\}$, so the inductive hypothesis implies that the latter coincides with the subfield generated by the image of $C[y_{h_{k+1}}]\{u\}$. 

Proof of Theorem 1. If the generic fiber of the projection of $Z$ to the $(y_{h+1}, u_{h+1})$-coordinates is of cardinality one, then every $\theta \in \Theta^\#$ is algebraic of degree one over $C[y_{h_{k+1}}, u_{h+1}]$ modulo $J_{h+1}$ (see [26, page 562]). Then the images of $\Theta^\#$ in $F$ belong to the subfield generated by the image of $C\{u\}[y_{h+e}]$ in $F$, so the belong to $E$. Proposition 1 implies that $\Theta^\#$ are globally identifiable.

Let $\Theta^\#$ be globally identifiable. Proposition 1 implies that the image of every $\theta \in \Theta^\#$ in $F$ belongs to $E$. Consider $\theta \in \Theta^\#$. Lemma 6 implies that $E$ is generated by the image of $C\{u\}[y_{h_{k+1}}]$. Hence, $\theta$ is algebraic of degree one over $C\{u\}[y_{h_{k+1}}]$ modulo $J_{h+1} \cap C\{u\}[y_{h_{k+1}}]$. Let $P_0$ be such a relation. Since $S_{h+1}$ is a triangular set, $J_{h+1} = (S_{h+1}) : Q^\infty$, and the ideal $J_{h+1}$ is prime, $S_{h+1}$ is a characteristic set of $J_{h+1}$ (see [13, Definitions 5.5 and 5.10, Theorem 5.13]).

Therefore, $P_0$ can be reduced to zero with respect to $S_{h+1}$. Consider $P_0$ as a polynomial in all derivatives of $u$ that do not belong to $u_{h+1}$, i.e. do not occur in $S_{h+1}$. Then each of these coefficients can also be reduced to zero with respect to $S_{h+1}$, and so each of these coefficients belongs to $J_{h+1}$, therefore, to $J_{h+1} \cap C[y_{h_{k+1}}, u_{h+1}]$. Taking one of them involving $\theta$, we obtain a polynomial equation of degree one for $\theta$ over $C[y_{h_{k+1}}, u_{h+1}]$ modulo $J_{h+1} \cap C[y_{h_{k+1}}, u_{h+1}]$. Since this holds for every $\theta \in \Theta^\#$, the cardinality of the generic fiber of the projection of $Z$ onto $(y_{h+1}, u_{h+1})$-coordinates is one.

The following statement can be proved in the same way as Theorem 1 using the part of the statement of Proposition 1 about local identifiability.

**Proposition 2.** We will use the notation from Theorem 1. Parameters $\Theta^\# \subset \Theta$ are locally identifiable if and only if the generic fiber of the projection of $Z$ to the $(y_{h+1}, u_{h+1})$-coordinates is finite.

**Proposition 3.** For all $h \in \mathbb{Z}_{\geq 0}^m$, the projection of $Z_h$ to the $(y_{h}, u_{h})$-coordinates is dominant if and only if the rank of the Jacobian of $S_h$ with respect to $(\mu, x_h)$ is equal to $|S_h|$ on a dense open subset of $Z_h$.

**Proof.** We set $N := |y_{h}| + |x_h| + |\mu| + |u_{h}|$ and $N_0 := |y_{h}| + |u_{h}|$. Then $Z_h \subset \mathbb{A}^N$, and we denote the projection to the $(y_{h}, u_{h})$-coordinates by $\pi : \mathbb{A}^N \to \mathbb{A}^N_0$. Let $M$ be the Jacobian matrix of $S_h$ with respect to $(x_h, \mu)$. Since $S_h$ is a triangular set and $J_h = (S_h) : Q^\infty$, codim $Z_h = |S_h|$ by [13, Theorem 4.4].

Assume that $\pi(Z_h)$ is not dense in $\mathbb{A}^{N_0}$. Then, by [28, Theorem 1.25], for every $x \in Z_h$,

$$\dim \pi^{-1}(\pi(x)) \cap Z_h > \dim Z_h - N_0.$$ (9)
Let \( p := (y_h^*, u_h^*) \in \pi(Z_h) \). Then
\[
M^* := M[y_h \leftarrow y_h^*, u_h \leftarrow u_h^*]
\]
can be viewed as the Jacobian of the polynomials \( S_h[y_h \leftarrow y_h^*, u_h \leftarrow u_h^*] \) in \((x_h, \mu)\), which all vanish on \( \pi^{-1}(p) \cap Z_h \). Then the rank of \( M^* \) at every point of \( \pi^{-1}(p) \cap Z_h \) does not exceed the codimension of \( \pi^{-1}(p) \cap Z_h \) in \( \pi^{-1}(p) \) [9, Theorem 16.19], which, by (9), is less than
\[
(N - N_0) - (\dim Z_h - N_0) = \text{codim} Z_h.
\]
Assume that \( \pi(Z_h) \) is dense in \( A^{N_0} \). Let \( A := \mathbb{C}[y_h, u_h] \) and \( B := \text{Quot}(A) \). Hence, \( J_h \cap A = 0 \). We introduce a new variable \( z \) and set \( \tilde{R} := R_h[z] \) and \( \tilde{J} := (S_h, Qz-1) \). Then \( \tilde{J} \cap R_h = J_h \) by [7, Theorem 14, page 205]. The ideal \( \tilde{J} \) is prime because \( \tilde{R}/\tilde{J} \) can be obtained from the domain \( R_h/J_h \) by inverting the image of \( Q \). Since \( \tilde{J} \cap A = \{0\} \), \( B \cdot \tilde{J} \) is a prime ideal in the \( B \)-algebra \( B \otimes_A \tilde{R} \). Note that \( (B \otimes_A \tilde{R}/B \cdot \tilde{J})_0 \) is a regular local ring. Let \( c \) denote the codimension of \( B \cdot \tilde{J} \) in \( B \otimes_A \tilde{R} \). [9, Corollary 16.20] implies the ideal of \( B \otimes_A \tilde{R}/B \cdot \tilde{J} \) generated by the \( c \times c \) minors of \( \tilde{M} \), the Jacobian of \( \{S_h, Qz-1\} \) with respect to \( (z, \mu, x_h) \), strictly contains \( \{0\} \). Therefore, \( \text{rank} \tilde{M} \geq c \). By [9, Theorem 16.19], \( \text{rank} \tilde{M} \leq c \). Since \( \tilde{M} \) is of the form
\[
\begin{pmatrix}
Q \\
0 & M
\end{pmatrix},
\]
its rank is equal to the rank of \( M \) modulo \( J_h \) plus one. Since \( \tilde{J} \cap A = \{0\} \), the codimension of \( B \cdot \tilde{J} \) in \( B \otimes_A \tilde{R} \) is equal to the codimension of \( \tilde{J} \) in \( \tilde{R} \), and the latter is equal to one plus the codimension of \( J_h \) in \( R_h \). Thus, the rank of \( M \) modulo \( J_h \) is equal to \( |S_h| \).

**Corollary 1.** There exists \( h \in \mathbb{Z}_{\geq 0}^m \) satisfying requirements (1) and (2) from Theorem 1. Moreover, for every \( h = (h_1, \ldots, h_m) \) satisfying requirement (1), \( h_1 + \ldots + h_m \leq s = |\Theta| \).

**Proof.** Let \( h = (h_1, \ldots, h_m) \) satisfy requirement (1) from Theorem 1. By Proposition 3, \( |S_h| \leq |x_h| + |\mu| \). Since also \( |S_h| \) is equal to the sum of \( |y_h| \) and the number of the elements of \( x_h \) of nonzero order,
\[
|S_h| \geq |y_h| + |x_h| - n = h_1 + \ldots + h_m + |x_h| - n.
\]
Therefore, \( h_1 + \ldots + h_m \leq n + |\mu| = s \). Thus, \( h_1 + \ldots + h_m \leq s \).

Let \( H \) be the set of all \( h \in \mathbb{Z}_{\geq 0}^m \) that satisfy requirement (1) from Theorem 1. Since \( h_1 + \ldots + h_m \leq s \) for every \( h = (h_1, \ldots, h_m) \in H \), \( H \) is finite. Then it contains a maximal element with respect to the coordinate-wise partial ordering. Such an element also satisfies requirement (2) from Theorem 1.
Below are several natural projections between them:

\[
\pi_{\text{io}} : V \to V_{\text{io}}, \quad \pi_{\#} : V \to V_{\#}, \quad \pi : V_{\#} \to V_{\text{io}}.
\]

Using this notation, we can define \( Z \) from the statement of Theorem 1 as \( Z = \pi_{\#}(Z_{h+1}) \).

**Theorem 2.** There exists a polynomial \( P \in \mathbb{C}[V_{\#}] \) such that

1. \( P \) does not vanish everywhere on \( Z \);
2. \( \deg P \leq (2 + |\theta_{\#}|) \deg Z_{h+1} \);
3. For all \( a \in Z \) such that \( P(a) \neq 0 \), the following statements are equivalent:
   - (a) every \( \theta \in \theta_{\#} \) is globally identifiable,
   - (b) \((J_{h+1} + I(\pi(a)) \cdot \mathbb{C}[V]) \cap \mathbb{C}[V_{\#}] = I(a)\),
   - (c) the zero set of \((J_{h+1} + I(\pi(a)) \cdot \mathbb{C}[V]) \cap \mathbb{C}[V_{\#}]\) is \( \{a\} \).

Before proving Theorem 2, we formulate and prove two technical lemmas, in which we do not use any notation introduced beyond Section 3.1.

**Lemma 7.** Let \( n, m, \) and \( r \) be non-negative integers such that \( n = m + r \), \( X \subset \mathbb{A}^n = \mathbb{A}^r \times \mathbb{A}^m \) an irreducible variety, and \( \pi : \mathbb{A}^n \to \mathbb{A}^m \) the projection onto the second component. Assume that the generic fiber of \( \pi|_X \) is finite.

1. Then there exists a proper subvariety \( Y \subset \overline{\pi(X)} \) such that
   - (a) \( \deg Y \leq \deg X \) and
   - (b) for every point \( p \in \overline{\pi(X)} \setminus Y \), there exists a closed (in the standard topology) ball \( B \subset \mathbb{A}^n \) centered at \( p \) such that \( \pi^{-1}(B) \cap X \) is compact (in the standard topology) and \( \pi^{-1}(p) \cap X \neq \emptyset \).

2. Then there exists a hypersurface \( H \subset \mathbb{A}^n \) not containing \( X \) such that
   - (a) \( \deg H \leq r \cdot (\deg X - 1) \) and
   - (b) for every \( p \in X \setminus H \), there exists a closed ball (in the standard topology) \( B \subset \mathbb{A}^m \) centered at \( \pi(p) \) such that \( \pi \) defines a bijection between the connected component (in the standard topology) of \( \pi^{-1}(B) \cap X \) containing \( p \) and its image.

3. If the generic fiber of \( \pi|_X \) is of cardinality one, then there exists a hypersurface \( H \subset \mathbb{A}^n \) not containing \( X \) such that
   - (a) \( \deg H \leq r \cdot \deg X \) and
   - (b) for every \( p \in X \setminus H \),
     \[
     I(X) + I(\pi(p)) \cdot \mathbb{C}[\mathbb{A}^n] = I(p).
     \]

**Proof.** We introduce coordinates \( \mathbf{x} := (x_1, \ldots, x_r) \) in \( \mathbb{A}^r \) and \( \mathbf{y} := (y_1, \ldots, y_m) \) in \( \mathbb{A}^m \). The condition that the generic fiber of \( \pi|_X \) is finite implies that \( \dim X = \dim \pi(X) \) [28, Theorem 1.25(ii)], so each element of \( \mathbf{x} \) is algebraic over \( \mathbb{C}[\mathbf{y}] \) modulo \( I(X) \). We prove each claim.
(1) Embed $\mathbb{A}^r$ into $\mathbb{P}^r$, then $\pi$ can be extended to $\pi_P: \mathbb{P}^r \times \mathbb{A}^m \to \mathbb{A}^m$. Let
\[
H_\infty := \mathbb{P}^r \times \mathbb{A}^m \setminus \mathbb{A}^r \times \mathbb{A}^m
\]
and $\overline{X}^\mathbb{P}$ be the closure of $X$ in $\mathbb{P}^r \times \mathbb{A}^m$. We set
\[
Y := \pi_\mathbb{P} \left( \overline{X}^\mathbb{P} \cap H_\infty \right).
\]
By [7, Corollary 9, page 431], $Y \subset \pi(X)$. Since $\pi|_X$ has finite generic fiber, $\dim X = \dim \pi(X)$ by [28, Theorem 1.25(ii)]. On the other hand,
\[
\dim Y \leq \dim \overline{X}^\mathbb{P} \cap H_\infty \leq \dim X - 1,
\]
hence $Y$ is a proper subvariety in $\pi(X)$. Also,
\[
\deg Y \leq \deg \overline{X}^\mathbb{P} \cap H_\infty \leq \deg X.
\]
Let $p \in \pi(X) \setminus Y$ and $B \subset \mathbb{A}^m$ a closed ball (of a positive radius) centered at $p$ such that $\pi^{-1}(B) \cap X$ is compact. Such a $B$ exists by [15, Lemma 2]. Moreover, $\pi^{-1}(B) \cap X \neq \emptyset$. Indeed, [21, Theorem 1, page 58] implies that $\pi(X)$ is dense in $\pi(X)$ with respect to the standard topology, so $B$ contains at least one point of $\pi(X)$. Let us show that $\pi^{-1}(p) \cap X \neq \emptyset$. For this, let $p_1, p_2, \ldots \in \pi(X) \cap B$ be a sequence of points converging to $p$, which exists because $p \in \pi(X)$. Let $q_1, q_2, \ldots \in \pi^{-1}(B) \cap X$ be such that $\pi(q_i) = p_i, i \geq 1$. Since $\pi^{-1}(B) \cap X$ is compact, there exists a converging subsequence of the sequence $q_1, q_2, \ldots$ with a limit $q \in X$. Since $p_1, p_2, \ldots$ converge to $p$ and $\pi$ is continuous, $\pi(q) = p$.

(2) We choose a subset $S \subset \{y\}$ that is a transcendence basis of $\mathbb{C}[y]$ modulo $I(X)$ over $\mathbb{C}$. For every $i, 1 \leq i \leq r$, the projection of $X$ to the $(x_i, S)$-coordinates is an irreducible hypersurface of degree at most $\deg X$. We denote its defining irreducible polynomial by $P_i(x_i, y)$. For every $i, 1 \leq i \leq r$, $\partial_{x_i} P_i$ does not vanish everywhere on $X$. Hence,
\[
P := \prod_{i=1}^r \partial_{x_i} P_i
\]
does not vanish everywhere on $X$. Let $H := Z(P)$. We prove that $H$ satisfies the requirements.

Consider $p \in X \setminus H$, and let $p = (x^*, y^*)$. Let $\tilde{X} := Z(P_1, \ldots, P_r)$, a subvariety in $\mathbb{A}^r \times \mathbb{A}^m$. Then $X \subset \tilde{X}$. Since $P(p) \neq 0$, the Jacobian of $P_1, \ldots, P_r$ with respect to $x$ is invertible at $p$. The implicit function theorem [25, Theorem 3.1.4] (applied to $X = \mathbb{A}^m$, $Y = \mathbb{A}^r$, $f = (P_1, \ldots, P_r)$) implies that there exist neighborhoods $U_1 \subset \mathbb{A}^m$ and $U_2 \subset \mathbb{A}^r$ of $y^*$ and $x^*$, respectively, such that $S(\tilde{X} \cap U_2 \times U_1)$ is a graph of some function $\varphi: U_1 \to U_2$, where $S(a, b) := (b, a), a \in \mathbb{A}^r, b \in \mathbb{A}^m$. Hence, $\pi$ defines a bijection between $\tilde{X} \cap U_2 \times U_1$ and its image under $\pi$.

Let $B \subset U_1$ be a closed ball centered at $\pi(p)$ and $C$ the connected component of $\tilde{X} \cap \pi^{-1}(B)$ containing $p$. Let $G_B$ denote the graph of $\varphi|_B$. We claim that $C \subset S(G_B)$. Consider the intersection
\[
C \cap S(G_B) = C \cap \left( \tilde{X} \cap (U_2 \times B) \right).
\]
Since $C \subset \tilde{X}$, the latter intersection is equal to $C \cap (U_2 \times B)$. Since $\pi(C) \subset B$, the latter intersection is the same as $C \cap (U_2 \times U_1)$. Hence,
\[
C \cap S(G_B) = C \cap (U_2 \times U_1).
\]

(10)
Since $S(G_B)$ is closed, $U_2 \times U_1$ is open, and $C$ is connected,

$$C \cap (U_2 \times U_1) = \emptyset \quad \text{or} \quad C \cap (U_2 \times U_1) = C.$$ 

Since $p \in C \cap (U_2 \times U_1)$, by (10), we have $C \cap S(G_B) = C$, and so $C \subset S(G_B)$. Since $S(G_B)$ maps bijectively onto $\pi(S(G_B))$, $C$ also maps bijectively onto $\pi(C)$. Since $X \subset X$, the connected component $C_0$ of $\pi^{-1}(B) \cap X$ containing $p$ is a subset of $C$, so $\pi$ defines a bijection between $C_0$ and $\pi(C_0)$.

(3) Since each element of $x$ is algebraic over $\mathbb{C}[y]$ modulo $I(X)$, there exists a representation of $I(X)$ as a triangular set $P_1, \ldots, P_r$ with respect to an ordering of the form

$$x_1 > x_2 > \ldots > x_r > y \quad \text{in some order}$$

so that $x_i$ is the leading variable of $P_i$, $1 \leq i \leq r$. Since the cardinality of the generic fiber is one, $\deg x_i P_i = 1$ for every $i$, $1 \leq i \leq r$ (see [26, page 562]). Since $P_i$ is reduced with respect to $P_{i+1}, \ldots, P_r$, $P_i$ does not depend on any variable in $x$ except for $x_i$. [8, Theorem 2] implies that the triangular set can be chosen in such a way that the degrees of the coefficients of $P_i$ as a polynomial with respect to $x_i$ do not exceed $\deg X$. We set $P$ to be the product of the leading coefficients of $P_1, \ldots, P_r$ and $H := Z(P)$ in $\mathbb{A}^n$. Then $\deg P \leq r \cdot \deg X$.

Let $p \in X \setminus H$. Then $\pi(p)$ is not a zero of $P$. We observe that $P_1, \ldots, P_r \in I(X)$ since $I(X)$ is a prime ideal. Then the ideal

$$J := I(X) + I(\pi(p)) \cdot \mathbb{C}[\mathbb{A}^n]$$

contains a polynomial $P_i(x, \pi(p)) = a_i x_i + b_i$, where $a_i \in \mathbb{C}^*$, $b_i \in \mathbb{C}$, for every $1 \leq i \leq r$. Since $p$ is a common zero of all polynomials in $J$, $\frac{-b_i}{a_i}$ is the value of the $x_i$-th coordinate of $p$. Since $I(\pi(p))$ also contains a polynomial of the form $y_j - c_j$, where $c_j \in \mathbb{C}$ is the value of the $y_j$-th coordinate of $p$, for every $j$, $1 \leq j \leq m$, the ideal $J$ is simply $I(p)$. \hfill $\square$

**Lemma 8.** Let $n$, $m$, and $r$ be non-negative integers such that $n = m + r$, $X \subset \mathbb{A}^n = \mathbb{A}^r \times \mathbb{A}^m$ an irreducible variety, and $\pi : \mathbb{A}^n \to \mathbb{A}^m$ the projection onto the second component. Then there exists a proper subvariety $Y \subset \overline{\pi(X)}$ such that

(1) $\deg Y \leq \deg X$ and

(2) for every $p \in \overline{\pi(X)} \setminus Y$, $\pi^{-1}(p) \cap X \neq \emptyset$.

**Proof.** If $\dim \pi(X) = 0$, then the irreducibility of $X$ implies that $\overline{\pi(X)}$ is a single point. Hence, we can choose $Y = \emptyset$. Before finishing the proof, we will first prove the following.

**Claim.** Assume that both the generic fiber of $\pi|_X$ and $\overline{\pi(X)}$ are not zero-dimensional. Then there exists a non-empty open subset $\mathcal{U}$ in the space of all hyperplanes in $\mathbb{A}^n$ such that, for all $H \in \mathcal{U}$,

- $X \not\subset H$,
- $H \cap X$ is irreducible, and
- $H \cap X$ projects dominantly onto $\pi(X)$.

**Proof.** In this case, $\dim X \geq 2$ by [28, Theorem 1.25(ii)], so Bertini’s theorem [2, III.7.(i)] implies that $H \cap X$ is irreducible for a generic $H$. Let $U$ be a non-empty open subset of $\overline{\pi(X)}$ such that, for every $p \in U$,

$$\dim \pi^{-1}(p) \cap X > 0.$$
Such a $U$ exists by [28, Theorem 1.25(ii)]. Since $U$ is dense in $\overline{\pi(X)}$, there exists a set of points $p_1, \ldots, p_N \in U$ for some $N$ such that, if a polynomial of degree at most $\deg X$ vanishes at $p_1, \ldots, p_N$, then it vanishes on $\overline{\pi(X)}$. Consider a hyperplane $H$ such that

1. $H \cap X$ is irreducible and
2. for every $i$, $1 \leq i \leq N$,
   \[ H \cap \pi^{-1}(p_i) \cap X \neq \emptyset. \]

Since conditions (1) and (2) are generic, the conjunction is also generic. Let $Y := \overline{\pi(H \cap X)}$, then $p_1, \ldots, p_N \in Y$. Since $\deg Y \leq \deg X$, [10, Proposition 3] implies that $Y$ can be defined by polynomials of degree at most $\deg X$. If $Y \subseteq \overline{\pi(X)}$, then there exists a polynomial of degree at most $\deg X$ that vanishes on $Y$ (and, in particular, at $p_1, \ldots, p_N$), but does not vanish on $\overline{\pi(X)}$. This is impossible. $\square$

We now return to the proof of Lemma 8. Assume that the generic fiber of $\pi|_X$ has dimension $d$. Applying the claim $d$ times, we obtain an affine subspace $L$ such that $X \cap L$ is irreducible, $X \cap L$ projects dominantly onto $\overline{\pi(X)}$, and the generic fiber of $\pi|_{X \cap L}$ is finite by [28, Theorem 1.25(ii)]. Applying statement (1) of Lemma 7 to $X \cap L$, we obtain a subvariety $Y \subset \overline{\pi(X)}$ of degree at most $\deg X$ such that every point in $\overline{\pi(X)} \setminus Y$ has a preimage in $X \cap L$. Then it has a preimage in $X$. $\square$

Proof of Theorem 2. Since $\theta^#$ are locally identifiable $\pi|_Z$ has finite generic fiber due to Proposition 2. We will construct a polynomial $P \in C[V_\#]$ as follows

- If $\theta^#$ is globally identifiable, then we set $P$ to be the product of the polynomials $P_1$ and $P_{\text{lift}}$ defined below.

  1. Applying statement (3) of Lemma 7 with $X = Z$ and $\pi = \pi$, we obtain a hypersurface $H \subset V_{\pi_0}$ of degree at most $|\theta^#| \deg Z$. We set $P_1$ to be the defining polynomial of $H$.

  2. Applying Lemma 8 to $X = Z_{h+1}$ and $\pi = \pi#$, we obtain a proper subvariety $Y_2 \subset Z$ of degree at most $\deg Z_{h+1}$. Since the generic fiber of $\pi|_Z$ is finite, [28, Theorem 1.25(ii)] implies that $\dim Z = \dim \pi(Z)$, so $\overline{\pi(Y_2)}$ is a proper subvariety of $\overline{\pi(Z)}$. [10, Proposition 3] implies that $\overline{\pi(Y_2)}$ can be defined by polynomials of degree at most $\deg Z_{h+1}$. We set $P_{\text{lift}}$ to be one of these polynomials that does not vanish everywhere on $Z$.

- If $\theta^#$ is not globally identifiable, then we set $P$ to be the product of the polynomial $P_{\text{lift}}$ defined above and the polynomials $P_{\infty}$ and $P_{\text{mult}}$ defined below.

  1. Applying statement (1) of Lemma 7 with $X = Z$ and $\pi = \pi$, we obtain a proper subvariety $Y_1 \subset \overline{\pi(Z)}$ of degree at most $\deg Z$. [10, Proposition 3] implies that $Y_1$ can be defined by polynomials of degree at most $\deg Z$. We set $P_{\infty}$ to be one of these polynomials that does not vanish everywhere on $\overline{\pi(Z)}$.

  2. Applying statement (2) of Lemma 7 with $X = Z$ and $\pi = \pi$, we obtain a hypersurface $H \subset V_{\#}$ of degree at most $|\theta^#| |(\deg Z - 1|$. We set $P_{\text{mult}}$ to be the irreducible defining polynomial of $H$.

Summing up the degree bounds, we obtain

\[ \deg P \leq \max(\deg(P_1 \cdot P_{\text{lift}}), \deg(P_{\infty} \cdot P_{\text{mult}} \cdot P_{\text{lift}})) \leq (2 + |\theta^#|) \cdot \deg Z_{h+1}. \]

In order to prove that $P$ satisfies requirement (3), we consider $a \in Z$ such that $P(a) \neq 0$. 

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To prove (3a) \(\implies\) (3b), assume that the parameters \(\theta^\#\) are globally identifiable. Since \(P_1(a) \neq 0\), the choice of \(P_1\) (see statement (3) of Lemma 7) implies that

\[
I(Z) + I(\pi(a)) \cdot \mathbb{C}[V_\#] = I(a).
\]  

(11)

Since \(I(Z) = I(J_{h+1}) \cap \mathbb{C}[V_\#]\),

\[
I(Z) + I(\pi(a)) \cdot \mathbb{C}[V_\#] \subset (J_{h+1} + I(\pi(a)) \cdot \mathbb{C}[V]) \cap \mathbb{C}[V_\#].
\]  

(12)

Since \(P_{\text{lift}}(a) \neq 0\),

\[
\pi^{-1}_\#(a) \cap Z_{h+1} \neq \emptyset.
\]

Hence, the ideal \(J_{h+1} + I(\pi(a)) \cdot \mathbb{C}[V]\) is proper, and so the right-hand side of (12) is a proper ideal of \(\mathbb{C}[V_\#]\). Since, by (11), it contains the maximal ideal \(I(a)\), it coincides with \(I(a)\).

The implication (3b) \(\implies\) (3c) follows from \(Z(I(a)) = \{a\}\).

To prove (3c) \(\implies\) (3a), we assume that the parameters \(\theta^\#\) are not globally identifiable. Denote the cardinality of the generic fiber of \(\pi|_Z\) by \(d > 1\). We define

\[
C(a) := \pi_\#(\pi_{io}^{-1}(\pi(a)) \cap Z_{h+1}).
\]

A direct computation shows that

\[
J := (J_{h+1} + I(\pi(a)) \cdot \mathbb{C}[V]) \cap \mathbb{C}[V_\#]
\]

vanishes at all the points of \(C(a)\). Thus, if we prove that \(|C(a)| > 1\), this would imply that the zero set of \(J\) in not \(\{a\}\).

Since \(P_{\text{lift}}(a) \neq 0\) and \(P \in C[V_{\text{io}}]\), for all \(b \in \pi^{-1}(\pi(a))\), \(P_{\text{lift}}(b) \neq 0\). Hence, the choice of \(P_{\text{lift}}\) implies (see Lemma 8) that, for all \(p \in \pi^{-1}(\pi(a)) \cap Z\),

\[
\pi^{-1}_\#(p) \cap Z_{h+1} \neq \emptyset.
\]  

(13)

Since \(\pi^{-1}_\#(p) \subset \pi_{io}^{-1}(\pi(p))\) and \(\pi(a) = \pi(p)\), we have \(\pi^{-1}_\#(p) \subset \pi_{io}^{-1}(\pi(a))\). Hence,

\[
\pi^{-1}_\#(p) \cap Z_{h+1} \subset \pi_{io}^{-1}(\pi(a)) \cap Z_{h+1}.
\]

Therefore, using (13),

\[
\pi^{-1}(\pi(a)) \cap Z \subset \pi_\#(\pi_{io}^{-1}(\pi(a)) \cap Z_{h+1}).
\]

Thus,

\[
|C(a)| \geq |\pi^{-1}(\pi(a)) \cap Z|.
\]

Let \(B_1\) and \(B_2\) be closed balls in \(V_{\text{io}}\) centered at \(\pi(a)\) such that

- \(\pi^{-1}(B_1) \cap Z\) is compact and
- \(\pi\) defines a bijection between the connected component of \(\pi^{-1}(B_2) \cap Z\) containing \(a\) and its image.

The existence of such \(B_1\) and \(B_2\) is implied by statements (1) and (2) of Lemma 7 because \(P_{\text{mult}}(a) \neq 0\) and \(P_{\text{nc}}(a) \neq 0\). We set \(B = B_1 \cap B_2\). Let \(C(B)\) be the set of connected components of \(\pi^{-1}(B) \cap Z\). Since \(\pi^{-1}(B) \cap Z\) is compact, the set \(C(B)\) is finite. Let \(D\) be the union of all \(C \in C(B)\) such that \(C \cap \pi^{-1}(\pi(a)) \cap Z = \emptyset\). Suppose that \(D \neq \emptyset\). Since \(D\) is compact, \(\pi(D)\) is compact, therefore, closed. Moreover, \(\pi(a) \notin \pi(D)\). Let \(B'\) be a closed ball centered at \(\pi(a)\) such that \(\pi(D) \cap B' = \emptyset\). We have \(B' \subset B\) and, for every \(C \in C(B')\),

\[
C \cap \pi^{-1}(\pi(a)) \cap Z \neq \emptyset.
\]
If \( D = \emptyset \), we set \( B' = B \). Assume that \( |\pi^{-1}(\pi(a)) \cap Z| = 1 \). Hence, \( \pi^{-1}(B') \cap Z \) has exactly one connected component. Therefore, for every \( p' \in \pi(Z) \cap B' \), we have \( |\pi^{-1}(p') \cap Z| = 1 \). Since \( \pi(Z) \cap B' \) is Zariski dense in \( \pi(Z) \), we arrive at a contradiction with the assumption that the generic fiber is of cardinality \( d' > 1 \). Hence, \( |C(a)| > 1 \).

**Notation 5.** Recall that \( Q \) is the common denominator of \( f \) and \( g \). Let \( d_0 = \max(\deg Qf, \deg Qg) \).

The following statement (with Theorem 2 and Proposition 3) is used in our design of Algorithm 1.

**Proposition 4.** For all \( h' \in \mathbb{Z}_{>0}^{m} \) and \( P \in R_{h'} \), such that \( P \) does not vanish everywhere on \( Z_{h'} \), there exists a polynomial \( \tilde{P} \in \mathbb{C}[\theta, u_{h'}] \) of degree at most \((1 + d_0 \cdot (2 \max h' - 1)) \cdot \deg P \) such that, for all \( \tilde{\theta} \) and \( \tilde{u}_{h'} \),

\[
\tilde{P}(\tilde{\theta}, \tilde{u}_{h'}) \neq 0 \implies P \text{ does not vanish at } \pi_{ip}^{-1}(\tilde{\theta}, \tilde{u}_{h'}) \cap Z_{h'},
\]

where \( \pi_{ip} \) is the projection from the space with coordinates \( \theta, x_{h'}, y_{h'}, u_{h'} \) to the space with coordinates \( \theta, u_{h'} = (\mu, x_1, \ldots, x_n, u_{h'}) \).

**Proof.** Consider the ranking on \( R_{h'} \) defined in the proof of Lemma 2. We will define a linear operator \( r: R_{h'} \to R_{h'} \). Consider a monomial \( m \in R_{h'} \). Let \( v \) be the leading variable of \( m \), so \( m = v \cdot \tilde{m} \). Then

\[
r(m) := \begin{cases} 
  (Qx_i^{(j+1)} - (Qx_i - F_i^{(j)}) \tilde{m}, & v = x_i^{(j+1)}, \\
  (Qy_j^{(j)} - (Qy_j - G_j^{(j)}) \tilde{m}, & v = y_j^{(j)}, \\
  Q \cdot \tilde{m}, & \text{otherwise.}
\end{cases}
\]

By the definition of \( r(P) \) and \( J_{h'} \),

\[
QP - r(P) \in J_{h'} \text{ for every } P \in R_{h'}.
\]

We introduce the following weight function \( w \) by

\[
\begin{align*}
  w(u^{(i)}) &= 0, & i & \geq 0, \\
  w(x_i^{(j)}) &= \max(0, 2i - 1), & 1 \leq j \leq n, i & \geq 0, \\
  w(y_j^{(i)}) &= 2i + 1, & 1 \leq j \leq m, i & \geq 0, \\
  w(\mu_i) &= 0, & \mu_i & \text{in } \mu,
\end{align*}
\]

( extending \( w \) multiplicatively to monomials and as the \( \max \) to sums of monomials). A direct computation shows that \( w(P) > w(r(P)) \) if \( r(P) \neq QP \). Thus, there exists a finite sequence \( P_0, \ldots, P_q \) such that

\[
P_0 = P, \quad P_{i+1} = r(P_i) \neq QP_i \text{ for all } 0 \leq i < q, \quad \text{and} \quad r(P_q) = QP_q.
\]

We set \( \bar{P} := P_q \). Since \( w(P) \leq \deg P \cdot (2 \cdot \max h' - 1) \), we have

\[
q \leq \deg P \cdot (2 \cdot \max h' - 1).
\]

Therefore,

\[
\deg \bar{P} \leq \deg P + \deg P \cdot d_0 \cdot (2 \cdot \max h' - 1) = (1 + d_0 \cdot (2 \cdot \max h' - 1)) \cdot \deg P.
\]

Since \( r(\bar{P}) = Q\bar{P} \), we have \( \bar{P} \in \mathbb{C}[\theta, u_{h'}] \). Due to (14), we have

\[
Q^q \cdot P - \bar{P} \in J_{h'}.
\]

Therefore, for all \( \tilde{\theta}, \tilde{u}_{h'} \), and \( p \in \pi_{ip}^{-1}(\tilde{\theta}, \tilde{u}_{h'}) \cap Z_{h'} \), we have

\[
\bar{P}(\tilde{\theta}, \tilde{u}_{h'}) = Q^q(\tilde{\theta}, \tilde{u}_{h'}) \cdot P(p).
\]

Hence, if \( \bar{P}(\tilde{\theta}, \tilde{u}_{h'}) \neq 0 \), then \( P(p) \neq 0 \).
5 Algorithm

Notation 6. For presentation purposes of Algorithm 1, we will use the following notation.

- $1 = (1, \ldots, 1)$, where the length will be clear from the context.
- $u = (u, u^{(1)}, \ldots, u^{(s)})$, where $s = |\theta|$.
- $z_{\gamma} = \{z_i^{(j)} \mid 1 \leq i \leq \ell, 0 \leq j < \gamma_i\}$ for $z = (z_1, \ldots, z_\ell)$ and $\gamma = (\gamma_1, \ldots, \gamma_\ell) \in \mathbb{Z}_{\geq 0}^\ell$.

Proposition 5. Algorithm 1 produces correct output with probability at least $p$.

Proof. In our probability analysis of random vectors $\hat{\theta}$ and $\hat{u}$ sampled in steps 2.c and 3.b, we will use the following observation. For all polynomials $P, Q \in \mathbb{C}[\theta, u]$ such that $Q \neq 0$ and for every sample set of vectors $(\hat{\theta}, \hat{u})$, the probability of $P(\hat{\theta}, \hat{u}) \neq 0$ is greater than or equal to the probability of $P(\hat{\theta}, \hat{u})Q(\hat{\theta}, \hat{u}) \neq 0$.

We claim that the value of $\beta$ right before Step 2.g will satisfy the requirements for $h$ of Theorem 1 with probability at least $\frac{1 - p}{2}$. We run Algorithm 1 replacing checking of the rank of the Jacobian in Step 2.f with checking the fact that $Z_{\beta+1_k}$ projects dominantly to the $(y_{\beta+1_k}, u_{\beta+1_k})$-coordinates. Denote the number of iterations of the while loop in Step 2.f by $q$. For all $i, 1 \leq i \leq q$, we denote the values of $\beta$ and $E^t$ after the $i$-th such iteration by $\beta_i$ and $E^t_i$, respectively, where $\beta_i := (\beta_{1i}, \ldots, \beta_{im})$. Let $\beta_0$ and $E^0_0$ denote the initial values. Due to the construction, $\beta_q$ satisfies the requirements for $h$ of Theorem 1.

Comparing Step 2.(f)ii of Algorithm 1 and Step (2) of Notation 3, we see that, for every $\beta$,

$$E^t \cup \{Y_{1,\beta_1}, \ldots, Y_{m,\beta_m}\} = S_{\beta+1} \cup \{X_{10}, \ldots, X_{n0}\}$$

This and Notation 3 imply that

$$E^t = S_{\beta} \cup \{X_{10}, \ldots, X_{n0}\} \cup \{X_{ij} \in S_{\beta+1} \mid x_i^{(j)} \text{ does not appear in } S_{\beta}, 0 < j, 1 \leq i \leq m\}. \quad (16)$$

Proposition 3 together with (16) imply that, for all $k, 1 \leq k \leq m$, if $Z_{\beta+1_k}$ does not project dominantly to the $(y_{\beta+1_k}, u_{\beta+1_k})$-coordinates, then the Jacobian condition of the while loop in Step 2.f is also false.

Let $P_1$ denote a maximal minor of the Jacobian of $S_{\beta_q}$ with respect to $(\mu, x_{\alpha_q})$ that is nonzero modulo $J_{\beta_q}$. Decomposition (16) implies that there exists a maximal minor in the Jacobian of $E^t_q$ with respect to $(\hat{\theta}, x_{\alpha_q})$ with the determinant equal to $Q^aP_1$ for some $a$. If $Q \cdot P_1$ does not vanish after the substitution

$$x_{\alpha_q} \leftarrow \hat{x}_{\alpha_q}, \ y_{\beta_q} \leftarrow \hat{y}_{\beta_q}, \ u \leftarrow \hat{u}, \ \theta \leftarrow \hat{\theta}, \quad (17)$$

then $\text{Jac}_{\theta, x_{\alpha_q}}(E^t_q)$ has full rank at $(\hat{\theta}, \hat{x}_{\alpha_q}, \hat{y}_{\beta_q})$. Since $E^t_i \subset E^t_q$ for every $1 \leq i \leq q$, the corresponding Jacobians of $E^t_1, \ldots, E^t_{q-1}$ also have full ranks at the corresponding points. Thus, with this choice of $(\hat{\theta}, \hat{u})$, the value of $\beta$ right before Step 2.g will be $\beta_q$ and, therefore, satisfy the requirements for $h$ of Theorem 1.

We will bound the probability of non-vanishing of $Q^aP_1$ after substitution (17). Let $\beta_q = (\beta_{q1}, \ldots, \beta_{qm})$. Corollary 1 implies that $\beta_{q1} + \ldots + \beta_{qm} \leq s$, so $|S_{\beta_q}| \leq s + ns$. Hence,

$$\deg P_1 \leq s(n+1)d_0.$$

Let $\bar{P}_1 \in \mathbb{C}[\theta, u]$ be such that,

- $\deg \bar{P}_1 \leq d_0 s(n+1)(1 + d_0(2s - 1))$ and
- for all $\hat{\theta}$ and $\hat{u}$, $\bar{P}_1(\hat{\theta}, \hat{u}) \neq 0$ implies that $P_1$ does not vanish at $\pi_{d_0}^{-1}(\hat{\theta}, \hat{u}) \cap Z_{\beta_q}$.
Algorithm 1: Global Identifiability

In $\Sigma$ : an algebraic differential model given by rational functions $f(x, \mu, u)$, and $g(x, \mu, u)$

$\theta^\ell$ : a subset of $\theta = \mu \cup x^*$ of locally identifiable parameters

$p$ : an element of $(0, 1)$

Out $\theta^g$ : the set of all globally identifiable parameters in $\theta^\ell$

the $\theta^g$ is correct with probability at least $p$

1. [Construct the maximal system $E$ of algebraic equations]
   (a) $s := |\theta|$, $Q :=$ the common denominator of $f$ and $g$
   (b) $X_N := x_i^{(0)} - x_i^*$ for $1 \leq i \leq n$
   (c) $X_{ij} := \langle Q \eta_i - Q f_j \rangle^{(j-1)}$ for $1 \leq i \leq n$ and $1 \leq j \leq s$
   (d) $Y_{ij} := \langle Q \eta_j - Q g_j \rangle^{(j)}$ for $1 \leq i \leq m$ and $0 \leq j \leq s$
   (e) $E :=$ the set of all $X_N$ and $Y_{ij}$ computed in Steps 1.b, 1.c, and 1.d.

2. [Truncate the system $E$, obtaining $E^t$]
   (a) $d_0 := \max(\deg f, \deg g, \deg Q)$
   (b) $D_1 := 2d_0s(n+1)(1+2d_0s)/(1-p)$
   (c) $\hat{\theta}, \hat{u} :=$ random vectors such that each coordinate is an integer from $[1, D_1]$ and $Q(\hat{\theta}, \hat{u}_0) \neq 0$
   (d) Find the unique solution of the triangular system $E[\theta \leftarrow \hat{\theta}, \ u \leftarrow \hat{u}]$ (in which each equation is linear in its leader) for all the variables.

Denote the $x_i^{(j)}$- and $y_i^{(j)}$-components of the solutions by $\hat{x}_{ij}$ and $\hat{y}_{ij}$, respectively.

(e) Initialize $\alpha := (1, \ldots, 1) \in \mathbb{Z}_{\geq 0}^\beta$, $\beta := (0, \ldots, 0) \in \mathbb{Z}_{\geq 0}^m$ and $E^t := \{X_0, \ldots, X_n\}$

(f) While there exists $k$ such that $\text{Jac}_{\theta, \alpha'}(E^t \cup \{Y_{k,\beta_k}\})$ at $(\hat{\theta}, \hat{x}_{\alpha}, \hat{y}_{\beta}, \hat{u})$ has full rank
   i. Add $Y_{k,\beta_k}$ to $E^t$ and then increment $\beta_k$
   ii. While there exists $x_i^{(j)}$ appearing in $E^t \cup \{Y_{k,\beta_k+1}, \ldots, Y_{m,\beta_m}\}$ but $X_{ij} \notin E^t$, add $X_{ij}$ to $E^t$
   iii. Set $\alpha_i := \max\left\{\text{ord}_{x_i} P \mid P \in E^t\right\} + 1$ for every $1 \leq i \leq n$

(g) Add $Y_{1,\beta_1}, \ldots, Y_{m,\beta_m}$ to $E^t$, increment $\beta_i$ for every $1 \leq i \leq m$

3. [Randomize some variables in $E^t$, obtaining $\hat{E}^t$]
   (a) $D_2 := 6|\theta|^t \left(\prod_{P \in E^t} \deg P\right)/(1+2d_0 \max(\beta)/(1-p))$
   (b) $\hat{\theta}, \hat{u} :=$ random vectors such that each coordinate is an integer from $[1, D_2]$ and $Q(\hat{\theta}, \hat{u}_0) \neq 0$
   (c) Find the unique solution of the triangular system $E^t[\theta \leftarrow \hat{\theta}, \ u \leftarrow \hat{u}]$ (in which each equation is linear in its leader) for all the variables.

Denote the $x_i^{(j)}$- and $y_i^{(j)}$-components of the solutions by $\hat{x}_{ij}$ and $\hat{y}_{ij}$, respectively.

(d) $\hat{E}^t := E^t[\hat{y}_{\beta} \leftarrow \hat{y}_{\beta}, \ u \leftarrow \hat{u}]$, $\hat{Q} := Q[u \leftarrow \hat{u}]$

4. [Determine $\theta^g$ from $\hat{E}^t$]
   (a) $\theta^g := \{\theta \in \theta^g \mid$ the system $\hat{E}^t = 0 \& \hat{Q} \neq 0 \& \theta \neq \hat{\theta}$ is inconsistent}
we obtain
\[ P \text{ applied to} \quad \text{then the decision of Algorithm 1 regarding global identifiability of} \]
\[ \text{The choice of} \quad \hat{\theta} \text{ of the zero set of} \quad \theta_1 \]
Consider \( i \leq i \leq |\theta^e| \). Let \( P_{2,i} \) be a polynomial whose existence is proven in Theorem 2 applied to \( \theta^e = (\theta_i^e) \). Consider \( a := (\hat{\mu}, \hat{x}_{\alpha_q}, \hat{y}_{\beta_q+1}, \hat{u}) \). Then (18) implies that the projection of the Zariski closure of the zero set of \( \hat{E}^i = 0 \) of \( \hat{Q} \neq 0 \) to the \( (\mu, x_{\alpha_q}, y_{\beta_q+1}, u) \)-coordinates is the zero set of the ideal
\[ J_{\beta_q+1} + I(\pi_{io}(a)) \cdot \mathbb{C}[\mu, x_{\alpha_q}, y_{\beta_q+1}, u]. \]
Hence, \( \theta^e_i \) will be added to \( \theta^g \) if and only if
\[ Z \left( \left( J_{\beta_q+1} + I(\pi_{io}(a)) \cdot \mathbb{C}[\mu, x_{\alpha_q}, y_{\beta_q+1}, u] \right) \cap \mathbb{C}[\theta^e_i] \right) = \{ \theta^e_i \}. \] (19)
The choice of \( P_{2,i} \) implies that (19) is equivalent to the fact that \( \theta^e_i \) is globally identifiable under the assumption that \( P_{2,i}(a) \neq 0 \). Thus, if
\[ P_{2,i}(a) \neq 0, \] (20)
then the decision of Algorithm 1 regarding global identifiability of \( \theta^e_i \) will be correct.
Let \( P_2 := \prod_{i=1}^{\theta^e} P_{2,i} \). Then \( P_2(a) \neq 0 \) implies that the output of Algorithm 1 is correct. Proposition 4 applied to \( \prod_{i=1}^{\theta^e} P_{2,i} \) provides a polynomial \( \hat{P}_2 \in \mathbb{C}[\theta, u] \) such that, for all \( \hat{\theta} \) and \( \hat{u} \), \( \hat{P}_2(\hat{\theta}, \hat{u}) \neq 0 \) implies that \( P_2 \) does not vanish at \( \pi_{ip}^{-1}(\hat{\theta}, \hat{u}) \cap Z_{\beta_q+1} \). Using the Bézout bound and the degree bound from Theorem 2, we obtain
\[ \deg Q \hat{P}_2 \leq 3|\theta^e| \deg Z_{\beta_q+1} \cdot (1 + 2d_0 \max \beta) \leq 3|\theta^e| \left( \prod_{P \in E^i} \deg P \right) (1 + 2d_0 \max \beta). \]
Since the coordinates of \( (\hat{\theta}, \hat{u}) \) are sampled from 1 to \( D_2 \), the Demillo-Lipton-Schwartz-Zippel lemma (see [36, Proposition 98]) implies that the point \( (\hat{\theta}, \hat{u}) \) is not a zero of \( Q \hat{P}_2 \) with probability at least
\[ 1 - \frac{\deg Q \hat{P}_2}{D_2} \geq \frac{1 + p}{2}. \]
Thus, with probability at least \( 2 \cdot \frac{1+p}{2} - 1 = p \), the output of Algorithm 1 is correct.

Remark 7. In Step 4 of Algorithm 1, we check the consistency of a system of equations and inequalities. This can be done (implemented) in many different ways. We list a few.
(1) Reformulate it into the following system of equations, via the Rabinowitsch trick,

\[ \tilde{E}^t \cup \{ z \cdot \hat{Q} - 1, w(\theta - \hat{\theta}) - 1 \}, \]

where \( z \) and \( w \) are new variables, and check its inconsistency. This can be done by using, for instance, Gröbner bases.

(2) Check

\[ \theta - \hat{\theta} \in \text{Ideal} \left( \tilde{E}^t \cup \{ z \cdot \hat{Q} - 1 \} \right). \]  

(21)

This can be done by using, for instance, Gröbner bases. Due to Theorem 2, condition (21) is equivalent to the consistency condition from Step 4. of Algorithm 1 under assumption (20) on the sampled point made in the proof of Proposition 5 for the index \( i \) of this particular \( \theta \).

(3) Check it directly using regular chains, which also allows inequations (see, e.g., [35, Section 2.2]). In practice, we observed that this method is generally slower for the problems that we considered than methods (1) and (2) are.

Example 5. We will illustrate the steps of the algorithm on a small input.

\[
\begin{align*}
\Sigma & := \left\{ x' = \frac{1}{x}, \\
y & = \mu_2 x + \mu_1, \\
x(0) & = x^*, \\
\theta^t & := \{ \mu_1, \mu_2, x^* \} \text{ (one can show that these parameters are locally identifiable)} \\
p & := 0.9
\end{align*}
\]

1. [Construct the maximal system \( E \) of algebraic equations]

(a) \( s := 3 \)

Since there is only one state variable and only one output variable, from now on, we will drop the first index from \( X \) and \( Y \) for brevity, for instance \( X_0 \) and \( Y_0 \) instead \( X_{1,0} \) and \( Y_{1,0} \), etc.

(b) \( X_0 := x - x^* \)

(c) \( X_1 := x x^{(1)} - 1, X_2 := x^2 x^{(2)} + x^{(1)}, X_3 := x^3 x^{(3)} + x x^{(2)} - 2(x^{(1)})^2 \)

(d) \( Y_0 := x y - \mu_2 x^2 - \mu_1 x, \ldots, Y_3 := x y^{(3)} - \mu_2 x x^{(3)} \)

(e) \( E := \{ X_0, \ldots, X_3, Y_0, \ldots, Y_3 \} \).

2. [Truncate the system \( E \), obtaining \( E^t \)]

(a) \( Q := x, d_0 := \text{deg}(\mu_2 x + \mu_1)x = 3. \)

(b) \( D_1 := 6840 \)

(c) \( (\hat{\mu}_1, \hat{\mu}_2, \hat{x}^*) := (1924, 2985, 1505) \)

(d) Solve the triangular system

\[
\begin{align*}
xy^{(3)} - 2985 xx^{(3)} &= 0, \\
xy^{(2)} - 2985 xx^{(2)} &= 0, \\
x^{(1)} - 2985 xx^{(1)} &= 0, \\
x y - 2985 x^2 - 1924 x &= 0, \\
x^3 x^{(3)} + (x^{(2)} x - 2(x^{(1)})^2) &= 0, \\
x^2 x^{(2)} + x^{(1)} &= 0, \\
x x^{(1)} - 1 &= 0, \\
x - 1505 &= 0.
\end{align*}
\]
3. \[\text{Randomize some variables in } E^t, \text{ obtaining } \hat{E}^t\]

(a) \(D_2 := 2216160\)

(b) \((\hat{\mu}_1, \hat{\mu}_2, \hat{x}^*) := (26723, 435749, 239812)\)

(c) We perform the same computation as in Step 2.d, but with the new numbers.

(d) \(\hat{E}^t := E^t \left[ y \leftrightarrow \hat{y}_0, y^{(1)} \leftrightarrow \hat{y}_1, y^{(2)} \leftrightarrow \hat{y}_2, y^{(3)} \leftrightarrow \hat{y}_3 \right], \quad \hat{Q} := x\)

4. \[\text{Determine } \theta^g \text{ from } \hat{E}^t\]

Following method (1) from Remark 7, we compute Gröbner bases for
\(\hat{E}^t \cup \{xz - 1, (\mu_1 - 26723)w - 1\}, \quad \hat{E}^t \cup \{xz - 1, (\mu_2 - 435749)w - 1\}, \quad \hat{E}^t \cup \{xz - 1, (x^* - 239812)w - 1\}\).

A computation shows that only the first Gröbner basis contains 1, so \(\mu_1\) is globally identifiable, and \(\mu_2\) and \(x^*\) are not.

Following method (2) from Remark 7, we compute the Gröbner basis of \(\hat{E}^t \cup \{xz - 1\}\) with respect to a degree reverse lexicographic ordering:

\[
3307376560508764078336x^{(3)} - 3z, \quad 57509795344x^{(2)} + z, \quad \mu_1 - 26723, \quad x^{(1)} - z, \\
57509795344z^2 - 1, \quad \mu_2 - 104497839188z, \quad x - 57509795344z, \quad x^* - x.
\]

Observe that the Gröbner basis does not contain \(x^* - 239812\) or \(\mu_2 - 435749\) but contains \(\mu_1 - 26723\).

Out \(\theta^g := \{\mu_1\}\)
6 Use with real-world systems

In this section, we describe the results of applying our implementation (run on MAPLE 2017) of Algorithm 1 to some systems of ODEs used in applications. The implementation and the examples are available at https://github.com/pogudingleb/Global_Identifiability. All running times were measured on a computer with 96 CPUs, 2.4 GHz each. We compare our results to the results obtained by DAISY [5] (version 1.9), which does not use parallel computations. Apart from the system itself, DAISY also takes as an input a positive integer SEED, which is used for sampling random points similarly to the numbers $D_1$ and $D_2$ in Algorithm 1. We are not aware of any bounds for the probability of the correctness of the output of DAISY in terms of the value of SEED, so we used the default value 35.

Example 6. The following system of ODEs corresponds to a chemical reaction network [6, Eq. 3.4], which is a reduced fully processive, $n$-site phosphorylation network.

$$\begin{align*}
\dot{x}_1 &= -\mu_1 x_1 x_2 + \mu_2 x_4 + \mu_4 x_6, \\
\dot{x}_2 &= -\mu_1 x_1 x_2 + \mu_2 x_4 + \mu_3 x_4, \\
\dot{x}_3 &= \mu_3 x_4 + \mu_5 x_6 - \mu_6 x_3 x_5, \\
\dot{x}_4 &= \mu_1 x_1 x_2 - \mu_2 x_4 - \mu_3 x_4, \\
\dot{x}_5 &= \mu_4 x_6 + \mu_5 x_6 - \mu_6 x_3 x_5, \\
\dot{x}_6 &= -\mu_4 x_6 - \mu_5 x_6 + \mu_6 x_3 x_5
\end{align*}$$

Setting outputs $y_1 = x_2$ and $y_2 = x_3$, we obtain a system of the form (1). We run Algorithm 1 setting $\mathbf{\theta}^I := \{\mu_1, \ldots, \mu_6, \ldots, x_1^*, \ldots, x_6^*\}$ (a calculation shows that these parameters are locally identifiable). The intermediate results are the following

- $D_1 = 4204800, D_2 = 4936445783 \cdot 10^{11}$;
- $\beta = (7, 7), \alpha = (6, 7, 6, 6, 6)$;
- the system $\tilde{E}^I$ consists of 52 equations in 50 variables.

The algorithm checks that all the parameters are globally identifiable with probability at least 99%. If we perform the last step of Algorithm 1 using method (1) from Remark 7, it takes 25 seconds (51 seconds of sequential computation). If we use method (2), it takes 25 seconds (55 seconds of sequential computation). DAISY did not output any result in 24 hours.

Example 7. The following version of SIWR is an extension of the SIR model, see [16, Eq. 3]:

$$\begin{align*}
\dot{s} &= \mu - \beta_I s i - \beta_W s w - \mu s + \alpha r, \\
i &= \beta_W s w + \beta_I s i - \gamma i - \mu i, \\
\dot{w} &= \xi (i - w), \\
\dot{r} &= \gamma i - \mu r - \alpha r
\end{align*}$$

where $s$, $i$, and $r$ stand for the fractions of the population that are susceptible, infectious, and recovered, respectively. The variable $w$ represents the concentration of the bacteria in the environment. The scalars $\alpha, \beta_I, \beta_W, \gamma, \mu, \xi$ are unknown parameters. Following [16], we assume that we can observe $y_1 = \kappa_1 i$, where $\kappa_1$ is one more unknown parameter. We will also assume that one can measure the total population $s + i + r$, so $y_2 = s + i + r$. We run Algorithm 1 setting $\mathbf{\theta}^I := \{\alpha, \beta_I, \beta_W, \gamma, \mu, \xi, \kappa_1, s^*, i^*, w^*, r^*\}$ (it was show in [16] that they are locally identifiable). The intermediate results are the following
\( D_1 = 2653200, \ D_2 = 1572960609 \cdot 10^{12}; \)
\( \beta = (10, 3), \ \alpha = (9, 10, 9, 8); \)
the system \( \hat{E}^t \) consists of 49 equations in 47 variables.

The algorithm checks that all the parameters are locally identifiable with probability at least 99\%. If we perform the last step of Algorithm 1 using method (1) from Remark 7, it takes 164 seconds (77 minutes of sequential computation). If we use method (2), it takes 14.5 minutes (114 minutes of sequential computation).

DAISY concluded that all parameters are likely to be globally identifiable, it took 30 minutes.

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