New order bounds in differential elimination algorithms[☆]

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Abstract

We present a new upper bound for the orders of derivatives in the Rosenfeld-Gröbner algorithm under weighted rankings. This algorithm computes a regular decomposition of a radical differential ideal in the ring of differential polynomials over a differential field of characteristic zero with an arbitrary number of commuting derivations. This decomposition can then be used to test for membership in the given radical differential ideal. In particular, this algorithm allows us to determine whether a system of polynomial PDEs is consistent.

In the case of one derivation, such a bound was given by Golubitsky, Kondratieva, Moreno Maza, and Ovchinnikov (2008). The only known bound in the case of several derivations was given by the authors of the present paper in 2016. The bound was achieved by associating to the algorithm antichain sequences whose lengths can be bounded using the results of León Sánchez and Ovchinnikov (2016). In the present paper, the above result by the current authors is generalized and significantly improved.

Keywords: Polynomial differential equations; differential elimination algorithms; computational complexity

1. Introduction

The Rosenfeld-Gröbner algorithm is a fundamental algorithm in the algebraic theory of differential equations. This algorithm, which first appeared in (Boulier et al., 1995, 2009), takes as

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its input a finite set F of differential polynomials and outputs a representation of the radical differential ideal generated by F as a finite intersection of regular differential ideals. The algorithm has many applications; for example, it can be used to test membership in a radical differential ideal, and, in conjunction with the differential Nullstellensatz, can be used to test the consistency of a system of polynomial differential equations. See (Golubitsky et al., 2008) for a history of the development of the Rosenfeld-Gröbner algorithm and similar decomposition algorithms.

The Rosenfeld-Gröbner algorithm has been implemented in Maple as a part of the `DifferentialAlgebra` package. In order to determine the complexity of the algorithm, we need to (among other things) find an upper bound on the orders of derivatives that appear in all intermediate steps and in the output of the algorithm. The first step in answering this question was completed in (Golubitsky et al., 2008), in which an upper bound in the case of a single derivation and any ranking on the set of derivatives was found. If there are n unknown functions and the order of the original system is h, the authors showed that an upper bound on the orders of the output of the Rosenfeld-Gröbner algorithm is $h(n-1)!$.

In this paper, we extend this result by finding an upper bound for the orders of derivatives that appear in the intermediate steps and in the output of the Rosenfeld-Gröbner algorithm in the case of an arbitrary number of commuting derivations and a weighted ranking on the derivatives. We first compute an upper bound for the weights of the derivatives involved for an arbitrary weighted ranking; by choosing a specific weight, we obtain an upper bound for the orders of the derivatives. For this, we construct special antichain sequences in the set $\mathbb{Z}^m_{\geqslant 0}\times\{1,\dots,n\}$ equipped with a specific partial order. A general analysis of lengths of antichain sequences in the context of PDEs began in (Pierce, 2014) and continued in (Freitag and León Sánchez, 2016) and (León Sánchez and Ovchinnikov, 2016). In the case of $m > 2$, we use results from (León Sánchez and Ovchinnikov, 2016) to estimate the lengths of our sequences. For $m = 2$, we exploit the fact that the constructed antichain has a special property, and this allows us to obtain better bounds.

We show that an upper bound for the weights of derivatives in the intermediate steps and in the output of the Rosenfeld-Gröbner algorithm is given by hf_{L+1}, where h is the weight of our input system of differential equations, $\{f_0, f_1, f_2, \dots\}$ is the Fibonacci sequence $\{0, 1, 1, 2, 3, 5, \dots\}$, and L is the maximal possible length of a certain antichain sequence (that depends solely on h, the number m of derivations, and the number n of unknown functions). Together with the bound for L, this gives us the final bound.

By choosing a specific weight, we are able to produce an upper bound for the orders of the derivatives in the intermediate steps and in the output of the Rosenfeld-Gröbner algorithm. Note that this bound is different from the upper bounds for the effective differential Nullstellensatz (D'Alfonso et al., 2014; Gustavson et al., 2016a), which are higher and also depend on the degree of the given system of differential equations. Our result is an improvement of (Gustavson et al., 2016b) because it allows us to compute sharper order upper bounds with respect to specific derivations than the previous upper bound did. In addition, our refinement in the case of $m = 2$ produces significantly improved upper bounds. For example, if $n = 2$ and $h = 3, 4, 5$, the new bound is approximately 11, 842, and 10^7 times better, respectively, than the upper bounds produced in (Gustavson et al., 2016b), see Section 6.1.

The paper is organized as follows. In Section 2, we present the background material from differential algebra that is necessary to understand the Rosenfeld-Gröbner algorithm. In Section 3, we describe this algorithm as it is presented in (Hubert, 2003), as well as two necessary auxiliary algorithms. In Section 4, we prove our main result on the upper bound. In Section 5, we prove our refined upper bound in the case of $m = 2$. In Section 6, we calculate the upper bound for specific values using the results of (León Sánchez and Ovchinnikov, 2016). In Section 7, we give an example showing that the lower bound for the orders of derivatives in the Rosenfeld-Gröbner algorithm is at least double-exponential in the number of derivations.

2. Background on differential algebra

In this section, we present background material from differential algebra that is pertinent to the Rosenfeld-Gröbner algorithm. For a more in-depth discussion, we refer the reader to (Hubert, 2003; Kolchin, 1973).

Definition 1. *A* differential ring *is a commutative ring R with a collection of m commuting derivations $\Delta = \{\partial_1, \ldots, \partial_m\}$ on R.*

Definition 2. *An ideal I of a differential ring is a* differential ideal *if $\delta a \in I$ for all $a \in I$, $\delta \in \Delta$.*

For a set $A \subseteq R$, let (A), $\sqrt{(A)}$, $[A]$, and $\{A\}$ denote the smallest ideal, radical ideal, differential ideal, and radical differential ideal containing A, respectively. If $\mathbb{Q} \subseteq R$, then $\{A\} = \sqrt{[A]}$.

Remark 3. *In this paper, as usual, we also use the braces $\{a_1, a_2, \ldots\}$ to denote the set containing the elements $a_1, a_2, \ldots$. Even though this notation conflicts with the above notation for radical differential ideals (used here for historical reasons), it will be clear from the context which of the two objects we mean in each particular situation.*

In this paper, $\mathbf{k}$ is a differential field of characteristic zero with m commuting derivations. The set of derivative operators is denoted by

$$\Theta := \left\{\partial_1^{i_1} \cdots \partial_m^{i_m} : i_j \in \mathbb{Z}_{\geqslant 0}, 1 \leqslant j \leqslant m\right\}.$$

For $Y = \{y_1, \ldots, y_n\}$ a set of n *differential indeterminates*, the set of derivatives of Y is

$$\Theta Y := \{\theta y : \theta \in \Theta, y \in Y\}.$$

Then the ring of *differential polynomials* over $\mathbf{k}$ is defined to be

$$\mathbf{k}\{Y\} = \mathbf{k}\{y_1, \ldots, y_n\} := \mathbf{k}[\theta y : \theta y \in \Theta Y].$$

We can naturally extend the derivations $\partial_1, \ldots, \partial_m$ to the ring $\mathbf{k}\{Y\}$ by defining

$$\partial_j \left(\partial_1^{i_1} \cdots \partial_m^{i_m} y_k\right) := \partial_1^{i_1} \cdots \partial_j^{i_j+1} \cdots \partial_m^{i_m} y_k.$$

For any $\theta = \partial_1^{i_1} \cdots \partial_m^{i_m} \in \Theta$, we define the *order* of θ to be

$$\mathrm{ord}(\theta) := i_1 + \cdots + i_m.$$

For any derivative $u = \theta y \in \Theta Y$, we define

$$\mathrm{ord}(u) := \mathrm{ord}(\theta).$$

For a differential polynomial $f \in \mathbf{k}\{Y\} \setminus \mathbf{k}$, we define the order of f to be the maximum order of all derivatives that appear in f. For any finite set $A \subseteq \mathbf{k}\{Y\} \setminus \mathbf{k}$, we set

$$\mathcal{H}(A) := \max\{\mathrm{ord}(f) : f \in A\}. \tag{1}$$

For any $\theta = \partial_1^{i_1} \cdots \partial_m^{i_m}$ and positive integers $c_1, \ldots, c_m \in \mathbb{Z}_{>0}$, we define the *weight* of θ to be

$$w(\theta) = w\left(\partial_1^{i_1} \cdots \partial_m^{i_m}\right) := c_1 i_1 + \cdots + c_m i_m.$$

Note that if all of the $c_i = 1$, then $w(\theta) = \mathrm{ord}(\theta)$ for all $\theta \in \Theta$. For a derivative $u = \theta y \in \Theta Y$, we define the weight of u to be $w(u) := w(\theta)$. For any differential polynomial $f \in \mathbf{k}\{Y\} \setminus \mathbf{k}$, we define the weight of f, $w(f)$, to be the maximum weight of all derivatives that appear in f. For any finite set $A \subseteq \mathbf{k}\{Y\} \setminus \mathbf{k}$, we set

$$\mathcal{W}(A) := \max\{w(f) : f \in A\}.$$

Definition 4. *A* ranking *on the set ΘY is a total order $<$ satisfying the following two additional properties: for all $u, v \in \Theta Y$ and all $\theta \in \Theta$, $\theta \neq \mathrm{id}$,*

$$u < \theta u \quad \text{and} \quad u < v \implies \theta u < \theta v.$$

A ranking $<$ is called an orderly ranking *if for all $u, v \in \Theta Y$,*

$$\mathrm{ord}(u) < \mathrm{ord}(v) \implies u < v.$$

Given a weight w, a ranking $<$ on ΘY is called a weighted ranking *if for all $u, v \in \Theta Y$,*

$$w(u) < w(v) \implies u < v.$$

Remark 5. *Note that if $w\left(\partial_1^{i_1} \cdots \partial_m^{i_m}\right) = i_1 + \cdots + i_m$ (that is, $w(\theta) = \mathrm{ord}(\theta)$), then a weighted ranking $<$ on ΘY is in fact an orderly ranking.*

From now on, we fix a weighted ranking $<$ on ΘY.

Definition 6. *Let $f \in \boldsymbol{k}\{Y\} \setminus \boldsymbol{k}$.*

- *The derivative $u \in \Theta Y$ of highest rank appearing in f is called the* leader *of f, denoted* $\mathrm{lead}(f)$.
- *If we write f as a univariate polynomial in* $\mathrm{lead}(f)$*, the leading coefficient is called the* initial *of f, denoted* $\mathrm{init}(f)$.
- *If we apply any derivative $\delta \in \Delta$ to f, the leader of δf is* $\delta(\mathrm{lead}(f))$*, and the initial of δf is called the* separant *of f, denoted* $\mathrm{sep}(f)$.

Given a set $A \subseteq \mathbf{k}\{Y\} \setminus \mathbf{k}$, we will denote the set of leaders of A by $\mathfrak{L}(A)$, the set of initials of A by I_A, and the set of separants of A by S_A; we then let $H_A = I_A \cup S_A$ be the set of initials and separants of A.

For a derivative $u \in \Theta Y$, we let $(\Theta Y)_{<u}$ (respectively, $(\Theta Y)_{\leqslant u}$) be the collection of all derivatives $v \in \Theta Y$ with $v < u$ (respectively, $v \leqslant u$). For any derivative $u \in \Theta Y$, we let $A_{<u}$ (respectively, $A_{\leqslant u}$) be the elements of A with leader $< u$ (respectively, $\leqslant u$), that is,

$$A_{<u} := A \cap \mathbf{k}[(\Theta Y)_{<u}] \quad \text{and} \quad A_{\leqslant u} := A \cap \mathbf{k}[(\Theta Y)_{\leqslant u}].$$

We can similarly define $(\Theta A)_{<u}$ and $(\Theta A)_{\leqslant u}$, where

$$\Theta A := \{\theta f : \theta \in \Theta, f \in A\}.$$

Given $f \in \mathbf{k}\{Y\} \setminus \mathbf{k}$ such that $\deg_{\mathrm{lead}(f)}(f) = d$, we define the *rank* of f to be

$$\mathrm{rank}(f) := \mathrm{lead}(f)^d.$$

The weighted ranking $<$ on ΘY determines a pre-order (that is, a relation satisfying all of the properties of an order, except for the property that $a \leqslant b$ and $b \leqslant a$ imply that $a = b$) on $\mathbf{k}\{Y\} \setminus \mathbf{k}$:

Definition 7. *Given $f_1, f_2 \in \boldsymbol{k}\{Y\} \setminus \boldsymbol{k}$, we say that*

$$\mathrm{rank}(f_1) < \mathrm{rank}(f_2)$$

if $\mathrm{lead}(f_1) < \mathrm{lead}(f_2)$ *or if* $\mathrm{lead}(f_1) = \mathrm{lead}(f_2)$ *and* $\deg_{\mathrm{lead}(f_1)}(f_1) < \deg_{\mathrm{lead}(f_2)}(f_2)$.

Definition 8. *A differential polynomial f is* partially reduced *with respect to another differential polynomial g if no proper derivative of* $\mathrm{lead}(g)$ *appears in f, and f is* reduced *with respect to g if, in addition,*

$$\deg_{\mathrm{lead}(g)}(f) < \deg_{\mathrm{lead}(g)}(g).$$

A differential polynomial is then (partially) reduced with respect to a set $A \subseteq \boldsymbol{k}\{Y\} \setminus \boldsymbol{k}$ if it is (partially) reduced with respect to every element of A.

Definition 9. *For a set $A \subseteq \boldsymbol{k}\{Y\} \setminus \boldsymbol{k}$, we say that A is:*

- autoreduced *if every element of A is reduced with respect to every other element.*
- weak d-triangular *if $\mathfrak{L}(A)$ is autoreduced.*
- d-triangular *if A is weak d-triangular and every element of A is partially reduced with respect to every other element.*

Note that every autoreduced set is d-triangular. Every weak d-triangular set (and thus every d-triangular and autoreduced set) is finite (Hubert, 2003, Proposition 3.9). Since the set of leaders of a weak d-triangular set A is autoreduced, distinct elements of A must have distinct leaders. If $u \in \Theta Y$ is the leader of some element of a weak d-triangular set A, we let A_u denote this element.

Definition 10. *We define a pre-order on the collection of all weak d-triangular sets, which we also call* rank, *as follows. Given two weak d-triangular sets $A = \{A_1, \ldots, A_r\}$ and $B = \{B_1, \ldots, B_s\}$, in each case arranged in increasing rank, we say that* $\mathrm{rank}(A) < \mathrm{rank}(B)$ *if either:*

- *there exists a $k \leqslant \min(r, s)$ such that* $\mathrm{rank}(A_i) = \mathrm{rank}(B_i)$ *for all $1 \leqslant i < k$ and* $\mathrm{rank}(A_k) < \mathrm{rank}(B_k)$*, or*
- *$r > s$ and* $\mathrm{rank}(A_i) = \mathrm{rank}(B_i)$ *for all $1 \leqslant i \leqslant s$.*

We also say that $\mathrm{rank}(A) = \mathrm{rank}(B)$ *if $r = s$ and* $\mathrm{rank}(A_i) = \mathrm{rank}(B_i)$ *for all $1 \leqslant i \leqslant r$.*

We can restrict this ranking to the collection of all d-triangular sets or the collection of all autoreduced sets.

Definition 11. *A* characteristic set *of a differential ideal I is an autoreduced set $C \subseteq I$ of minimal rank among all autoreduced subsets of I.*

Given a finite set $S \subseteq \mathbf{k}\{Y\}$, let S^∞ denote the multiplicative set containing 1 and generated by S. For an ideal $I \subseteq \mathbf{k}\{Y\}$, we define the *colon ideal* to be

$$I : S^\infty := \{a \in \mathbf{k}\{Y\} : \exists s \in S^\infty \text{ with } sa \in I\}.$$

If I is a differential ideal, then $I : S^\infty$ is also a differential ideal (Kolchin, 1973, Section I.2).

Definition 12. *For a differential polynomial $f \in \mathbf{k}\{Y\}$ and a weak d-triangular set $A \subseteq \mathbf{k}\{Y\}$, a* differential partial remainder *f_1 and a* differential remainder *f_2 of f with respect to A are differential polynomials such that there exist $s \in S_A^\infty$, $h \in H_A^\infty$ such that $sf \equiv f_1 \mod [A]$ and $hf \equiv f_2 \mod [A]$, with f_1 partially reduced with respect to A and f_2 reduced with respect to A.*

We denote a differential partial remainder of f with respect to A by $\text{pd-red}(f, A)$ and a differential remainder of f with respect to A by $\text{d-red}(f, A)$. There are algorithms to compute $\text{pd-red}(f, A)$ and $\text{d-red}(f, A)$ for any f and A (Hubert, 2003, Algorithms 3.12 and 3.13). These algorithms have the property that

$$\operatorname{rank}(\text{pd-red}(f, A)),\ \operatorname{rank}(\text{d-red}(f, A)) \leqslant \operatorname{rank}(f);$$

since we have a weighted ranking, this implies that

$$w(\text{pd-red}(f, A)),\ w(\text{d-red}(f, A)) \leqslant w(f).$$

Definition 13. *Two derivatives $u, v \in \Theta Y$ are said to have a* common derivative *if there exist $\phi, \psi \in \Theta$ such that $\phi u = \psi v$. Note this is the case precisely when $u = \theta_1 y$ and $v = \theta_2 y$ for some $y \in Y$ and $\theta_1, \theta_2 \in \Theta$.*

Definition 14. *If $u = \partial_1^{i_1} \cdots \partial_m^{i_m} y$ and $v = \partial_1^{j_1} \cdots \partial_m^{j_m} y$ for some $y \in Y$, we define the* least common derivative *of u and v, denoted $\operatorname{lcd}(u, v)$, to be*

$$\operatorname{lcd}(u, v) = \partial_1^{\max(i_1, j_1)} \cdots \partial_m^{\max(i_m, j_m)} y,$$

otherwise we set $\operatorname{lcd}(u, v) = 0$ and, for any weight w, define $w(0) = -1$.

Definition 15. *For $f, g \in \mathbf{k}\{Y\} \setminus \mathbf{k}$, we define the* Δ-polynomial *of f and g, denoted $\Delta(f, g)$, as follows. If $\operatorname{lead}(f)$ and $\operatorname{lead}(g)$ have no common derivatives, set $\Delta(f, g) = 0$. Otherwise, let $\phi, \psi \in \Theta$ be such that*

$$\operatorname{lcd}(\operatorname{lead}(f), \operatorname{lead}(g)) = \phi(\operatorname{lead}(f)) = \psi(\operatorname{lead}(g)),$$

and define

$$\Delta(f, g) := \operatorname{sep}(g)\phi(f) - \operatorname{sep}(f)\psi(g).$$

Definition 16. *A pair (A, H) is called a* regular differential system *if:*

- *A is a d-triangular set*
- *H is a set of differential polynomials that are all partially reduced with respect to A*
- *$S_A \subseteq H^\infty$*

- *for all $f, g \in A$, $\Delta(f, g) \in ((\Theta A)_{<u}) : H^\infty$, where $u = \operatorname{lcd}(\operatorname{lead}(f), \operatorname{lead}(g))$.*

Definition 17. *Any ideal of the form $[A] : H^\infty$, where (A, H) is a regular differential system, is called a* regular differential ideal*.*

Every regular differential ideal is a radical differential ideal (Hubert, 2003, Theorem 4.12).

Definition 18. *Given a radical differential ideal $I \subseteq \boldsymbol{k}\{Y\}$, a* regular decomposition *of I is a finite collection of regular differential systems $\{(A_1, H_1), \ldots, (A_r, H_r)\}$ such that*

$$I = \bigcap_{i=1}^{r} [A_i] : H_i^\infty.$$

Due to the Rosenfeld-Gröbner algorithm, every radical differential ideal in $\mathbf{k}\{Y\}$ has a regular decomposition.

Definition 19. *A d-triangular set C is called a* differential regular chain *if it is a characteristic set of $[C] : H_C^\infty$; in this case, we call $[C] : H_C^\infty$ a* characterizable differential ideal*.*

Definition 20. *A* characteristic decomposition *of a radical differential ideal $I \subseteq \boldsymbol{k}\{Y\}$ is a representation of I as an intersection of characterizable differential ideals.*

As we will recall in Section 3, every radical differential ideal also has a characteristic decomposition.

3. Rosenfeld-Gröbner algorithm

Below we reproduce the Rosenfeld-Gröbner algorithm from (Hubert, 2003, Section 6). This algorithm relies on two others, called auto-partial-reduce and update, which we also include. We include these two auxiliary algorithms because, in Section 4, we will study their effect on the growth of the weights of derivatives in Rosenfeld-Gröbner.

Rosenfeld-Gröbner takes as its input two finite subsets $F, K \in \mathbf{k}\{Y\}$ and outputs a finite set $\mathcal{A}$ of regular differential systems such that

$$\{F\} : K^\infty = \bigcap_{(A,H)\in\mathcal{A}} [A] : H^\infty, \tag{2}$$

where $\mathcal{A} = \varnothing$ if $1 \in \{F\} : K^\infty$.

If we have a decomposition of $\{F\} : K^\infty$ as in (2), we can compute, using only algebraic operations, a decomposition of the form

$$\{F\} : K^\infty = \bigcap_{C\in\mathcal{C}} [C] : H_C^\infty, \tag{3}$$

where $\mathcal{C}$ is finite and each $C \in \mathcal{C}$ is a differential regular chain (Hubert, 2003, Algorithms 7.1 and 7.2). This means that an upper bound on $\bigcup_{(A,H)\in\mathcal{A}} \mathcal{W}(A \cup H)$ from (2) will also be an upper bound on $\bigcup_{C\in\mathcal{C}} \mathcal{W}(C)$ from (3).

Rosenfeld-Gröbner has many immediate applications. For example, if $K = \{1\}$, then $\{F\} : K^\infty = \{F\}$, so in this case, Rosenfeld-Gröbner computes a regular decomposition of $\{F\}$, which then also gives us a characteristic decomposition of $\{F\}$ by the discussion in the previous paragraph.

The *weak differential Nullstellensatz* says that a system of polynomial differential equations $F = 0$ is consistent (that is, has a solution in some differential field extension of $\mathbf{k}$) if and only if $1 \notin [F]$ (Kolchin, 1973, Section IV.2). Thus, since $\mathsf{Rosenfeld\text{-}Gröbner}(F, K) = \varnothing$ if and only if $1 \in \{F\} : K^\infty$, we see that $F = 0$ is consistent if and only if $\mathsf{Rosenfeld\text{-}Gröbner}(F, \{1\}) \neq \varnothing$.

More generally, Rosenfeld-Gröbner and its extension for computing a characteristic decomposition of a radical differential ideal allow us to test for membership in a radical differential ideal, as follows. Suppose we have computed a characteristic decomposition

$$\{F\} = \bigcap_{C \in \mathcal{C}} [C] : H_C^\infty.$$

Now, a differential polynomial $f \in \mathbf{k}\{Y\}$ is contained in $\{F\}$ if and only if $f \in [C] : H_C^\infty$ for all $C \in \mathcal{C}$; this latter case is true if and only if $\text{d-red}(f, C) = 0$, which can be tested using (Hubert, 2003, Algorithm 3.13).

Rosenfeld-Gröbner, auto-partial-reduce, and update rely on the following tuples of differential polynomials:

Definition 21. *A* Rosenfeld-Gröbner quadruple *(or* RG-quadruple*) is a 4-tuple (G, D, A, H) of finite subsets of $\boldsymbol{k}\{Y\}$ such that:*

- *A is a weak d-triangular set, $H_A \subseteq H$, D is a set of Δ-polynomials, and*
- *for all $f, g \in A$, either $\Delta(f, g) = 0$ or $\Delta(f, g) \in D$ or*

  $$\Delta(f, g) \in (\Theta(A \cup G)_{<u}) : H_u^\infty,$$

  where $u = \operatorname{lcd}(\operatorname{lead}(f), \operatorname{lead}(g))$ and $H_u = H_{A_{<u}} \cup (H \setminus H_A) \cap \boldsymbol{k}[(\Theta Y)_{<u}]$.

Remark 22. *The RG-quadruple that is output by* update *satisfies additional properties that we do not list, as they are not important for our analysis. For more information, we refer the reader to (Hubert, 2003, Algorithm 6.10)*

4. Order upper bound for arbitrary m

Given finite subsets $F, K \subseteq \mathbf{k}\{Y\}$, let

$$h = \mathcal{W}(F \cup K). \tag{4}$$

Our goal is to find an upper bound for

$$\mathcal{W}\left(\bigcup_{(A,H) \in \mathcal{A}} (A \cup H)\right),$$

where $\mathcal{A} = \mathsf{Rosenfeld\text{-}Gröbner}(F, K)$, in terms of h, m (the number of derivations), and n (the number of differential indeterminates). By then choosing a specific weight, we can find an upper bound for $\mathcal{H}\left(\bigcup_{(A,H) \in \mathcal{A}} (A \cup H)\right)$ in terms of m, n, and $\mathcal{H}(F \cup K)$.

We approach this problem as follows. Every $(A, H) \in \mathcal{A}$ is formed by applying auto-partial-reduce to a 4-tuple $(\varnothing, \varnothing, A', H') \in \mathcal{S}$. Thus, it suffices:

Algorithm: Rosenfeld-Gröbner, (Hubert, 2003, Algorithm 6.11)
Data: F, K finite subsets of $\mathbf{k}\{Y\}$
Result: A set $\mathcal{A}$ of regular differential systems such that:

- $\mathcal{A}$ is empty if it has been detected that $1 \in \{F\} : K^\infty$
- $\{F\} : K^\infty = \bigcap_{(A,H)\in\mathcal{A}} [A] : H^\infty$ otherwise

$\mathcal{S} := \{(F, \varnothing, \varnothing, K)\}$;
$\mathcal{A} := \varnothing$;
while $\mathcal{S} \neq \varnothing$ **do**
  $(G, D, A, H) :=$ an element of $\mathcal{S}$;
  $\bar{\mathcal{S}} = \mathcal{S} \setminus (G, D, A, H)$;
  if $G \cup D = \varnothing$ **then**
    $\mathcal{A} := \mathcal{A} \cup$ auto-partial-reduce(A, H);
  else
    $p :=$ an element of $G \cup D$;
    $\bar{G}, \bar{D} := G \setminus \{p\}, D \setminus \{p\}$;
    $\bar{p} :=$ d-red(p, A);
    if $\bar{p} = 0$ **then**
      $\bar{\mathcal{S}} := \bar{\mathcal{S}} \cup \{(\bar{G}, \bar{D}, A, H)\}$;
    else
      if $\bar{p} \notin k$ **then**
        $\bar{p}_i := \bar{p} - \mathrm{init}(\bar{p})\,\mathrm{rank}(\bar{p})$ $\bar{p}_s := \deg_{\mathrm{lead}(\bar{p})}(\bar{p})\bar{p} - \mathrm{lead}(\bar{p})\,\mathrm{sep}(\bar{p})$;
        $\bar{\mathcal{S}} := \bar{\mathcal{S}} \cup \{$update$(\bar{G}, \bar{D}, A, H, \bar{p}), (\bar{G} \cup \{\bar{p}_s, \mathrm{sep}(\bar{p})\}, \bar{D}, A, H \cup \{\mathrm{init}(\bar{p})\}), (\bar{G} \cup \{\bar{p}_i, \mathrm{init}(\bar{p})\}, \bar{D}, A, H)\}$;
      end
    end
  end
  $\mathcal{S} := \bar{\mathcal{S}}$;
end
return $\mathcal{A}$;

Algorithm: auto-partial-reduce, (Hubert, 2003, Algorithm 6.8)

Data: Two finite subsets A, H of $\mathbf{k}\{Y\}$ such that $(\varnothing, \varnothing, A, H)$ is an RG-quadruple

Result:

- The empty set if it is detected that $1 \in [A] : H^\infty$
- Otherwise, a set with a single regular differential system (B, K) with $\mathfrak{L}(A) = \mathfrak{L}(B)$, $H_B \subseteq K$, and $[A] : H^\infty = [B] : K^\infty$

$B := \varnothing$;
for $u \in \mathfrak{L}(A)$ *increasingly* **do**
  $b :=$ pd-red(A_u, B);
  if rank$(b) =$ rank(A_u) **then**
    $B := B \cup \{b\}$;
  else
    return $(\varnothing)$;
  end
end
$K := H_B \cup \{\text{pd-red}(p, B) : p \in H \setminus H_A\}$;
if $0 \in K$ **then**
  return $(\varnothing)$;
else
  return $\{(B, K)\}$;
end

Algorithm: update (Hubert, 2003, Algorithm 6.10)

Data:

- A 4-tuple (G, D, A, H) of finite subsets of $\mathbf{k}\{Y\}$
- A differential polynomial p reduced with respect to A such that $(G \cup \{p\}, D, A, H)$ is an RG-quadruple

Result: A new RG-quadruple $(\bar{G}, \bar{D}, \bar{A}, \bar{H})$
$u :=$ lead(p);
$G_A := \{a \in A \mid \text{lead}(a) \in \Theta u\}$;
$\bar{A} := A \setminus G_A$;
$\bar{G} := G \cup G_A$;
$\bar{D} := D \cup \{\Delta(p, a) \mid a \in \bar{A}\} \setminus \{0\}$;
$\bar{H} := H \cup \{\text{sep}(p), \text{init}(p)\}$;
return $(\bar{G}, \bar{D}, \bar{A} \cup \{p\}, \bar{H})$;

- to bound how auto-partial-reduce increases the weight of a collection of differential polynomials (it turns out to not increase the weight), and
- to bound $\mathcal{W}(G \cup D \cup A \cup H)$ for all (G, D, A, H) added to $\mathcal{S}$ throughout the course of Rosenfeld-Gröbner.

We accomplish the latter by determining when the weight of a tuple (G, D, A, H) added to $\mathcal{S}$ is larger than the weights of the previous elements of $\mathcal{S}$ and bounding $\mathcal{W}(G \cup D \cup A \cup H)$ in this instance, and then bounding the number of times we can add such elements to $\mathcal{S}$.

There is a sequence (G_i, D_i, A_i, H_i) corresponding to each regular differential system (A, H) in the output of Rosenfeld-Gröbner, where $N = N_{(A,H)}$, such that $(G_{i+1}, D_{i+1}, A_{i+1}, H_{i+1})$ is obtained from (G_i, D_i, A_i, H_i) during the while loop, $(G_0, D_0, A_0, H_0) = (F, \varnothing, \varnothing, K)$, and $(A, H) =$ auto-partial-reduce(A_N, H_N).

We begin with an auxiliary result, which is an analogue of the *first* property from (Golubitsky et al., 2009, Section 5.1).

Lemma 23. *For every $f \in A_i$ and $i < j$, there exists $g \in A_j$ such that* $\operatorname{lead}(f) \in \Theta \operatorname{lead}(g)$. *In particular, if p is reduced with respect to A_j, then p is reduced with respect to A_i for all $i < j$.*

Proof. It is sufficient to consider the case $j = i + 1$. If $(G_{i+1}, D_{i+1}, A_{i+1}, H_{i+1})$ was obtained from (G_i, D_i, A_i, H_i) without applying update, then $A_i = A_{i+1}$. Otherwise, either $f \in A_i \setminus G_{A_i}$ (we use the notation from update), or $f \in G_{A_i}$. In the former case, $f \in A_{i+1}$ as well, so we can set $g = f$. In the latter case, $\operatorname{lead}(f) \in \Theta \operatorname{lead}(p)$, so we can set $g = p$. □

We define a partial order $\preccurlyeq$ on the set of derivatives ΘY as follows. For $u, v \in \Theta Y$, we say that $u \preccurlyeq v$ if there exists $\theta \in \Theta$ such that $\theta u = v$. Note that this implies that u and v are both derivatives of the same $y \in Y$.

Definition 24. *An* antichain sequence *in ΘY is a sequence of elements $S = \{s_1, s_2, \ldots\} \subseteq \Theta Y$ that are pairwise incomparable in this partial order.*

Given a sequence $\{(G_i, D_i, A_i, H_i)\}_{i=0}^{N}$ as above (where $N = N_{(A,H)}$ for some regular differential system (A, H) in the output of Rosenfeld-Gröbner), we will construct an antichain sequence $S = \{s_1, s_2, \ldots\} \subseteq \Theta Y$ inductively going along the sequence $\{(G_i, D_i, A_i, H_i)\}$. Suppose $S_{j-1} = \{s_1, \ldots, s_{j-1}\}$ has been constructed after considering $(G_0, D_0, A_0, H_0), \ldots, (G_{i-1}, D_{i-1}, A_{i-1}, H_{i-1})$, where $S_0 = \varnothing$. A 4-tuple (G_i, D_i, A_i, H_i) can be obtained from the tuple $(G_{i-1}, D_{i-1}, A_{i-1}, H_{i-1})$ in two ways:

(1) We did not perform update. In this case, we do not append a new element to S.

(2) We performed update with respect to a differential polynomial $\bar{p}$. If there exists $s_k \in S_{j-1}$ such that $\operatorname{lead}(\bar{p}) \preccurlyeq s_k$, we do not append a new element to S_{j-1}. Otherwise, let $s_j = \operatorname{lead}(\bar{p})$ and define $S_j = \{s_1, \ldots, s_j\}$. In the latter case, we set $k_j = i$. We also set $k_0 = 0$.

Lemma 25. *The sequence $\{s_j\}$ constructed above is an antichain in ΘY and, for all $j \geqslant 1$,*

$$w(s_j) \leqslant \max\left(h, \max_{l<k<j} w\left(\operatorname{lcd}(s_l, s_k)\right)\right), \tag{5}$$

where h is defined in (4).

Proof. Let $i < j$. Assume that $s_j \succcurlyeq s_i$. Then, p is not reduced with respect to A_{k_i}, which contradicts Lemma 23. On the other hand, the case $s_j \preccurlyeq s_i$ is impossible by the construction of the sequence, so $\{s_j\}$ is an antichain sequence.

We denote the maximal $j \in \mathbb{Z}_{\geqslant 0}$ such that $k_j \leqslant i$ by anti-k_i. For every $j \geqslant 1$, we set

$$F_j = \max\left(h, \max_{l<k<j} w\,(\operatorname{lcd}(s_l, s_k))\right),$$

and $F_0 = 0$. For all $i \geqslant 0$, let us set $j =$ anti-k_i and prove by induction on i that

(1) $\mathcal{W}\left(\bigcup_{t=0}^{i}(G_t \cup D_t \cup H_t)\right) \leqslant F_{j+1}$;

(2) $\mathcal{W}\left(\bigcup_{t=0}^{i} A_t\right) \leqslant F_j$;

(3) For every $g \in \bigcup_{t=0}^{i} A_t$, there exists $j_0 \leqslant j$ such that $s_{j_0} \in \Theta\,\operatorname{lead}(g)$.

In the base case $i = 0 = k_0$, we have

$$\mathcal{W}(G_0 \cup D_0 \cup H_0) = h = F_1 \quad \text{and} \quad \mathcal{W}(A_0) = \mathcal{W}(\varnothing) = 0 = F_0.$$

There are two distinct cases for $i + 1$:

(1) Case $i + 1 < k_{j+1}$ (so anti-$k_{i+1} = j$). Then, $(G_{i+1}, D_{i+1}, A_{i+1}, H_{i+1})$ was obtained from (G_i, D_i, A_i, H_i) in one of the following ways:

(a) We did not perform update. In this case, $A_{i+1} = A_i$ and

$$\mathcal{W}(G_{i+1} \cup D_{i+1} \cup H_{i+1}) \leqslant \mathcal{W}(G_i \cup D_i \cup H_i).$$

(b) We performed update with respect to a differential polynomial p such that $\operatorname{lead}(g) \in \Theta\,\operatorname{lead}(p)$ for some $g \in \bigcup_{t=0}^{i} A_t$. In this case,

$$\mathcal{W}(A_{i+1}) \leqslant \mathcal{W}\left(\bigcup_{t=0}^{i} A_t\right).$$

Moreover, due to the third inductive hypothesis, there exists $j_0 \leqslant j$ such that $s_{j_0} \in \Theta\,\operatorname{lead}(g) \subset \Theta\,\operatorname{lead}(p)$, and this proves the third hypothesis for $i + 1$. Then, for all $q \in A_t$ ($t \leqslant i$) with $s_{j_1} \in \Theta\,\operatorname{lead}(q)$ for some $j_1 \leqslant j$,

$$w(\Delta(p, g)) \leqslant w(\operatorname{lcd}(\operatorname{lead}(g), \operatorname{lead}(q))) \leqslant w(\operatorname{lcd}(s_{j_0}, s_{j_1})) \leqslant F_{j+1}.$$

Since $D_{i+1} \setminus D_i$ consists of some of these polynomials, $G_{i+1} \setminus G_i \subseteq A_i$, and $H_{i+1} \setminus H_i = \{\operatorname{sep}(p), \operatorname{init}(p)\}$, then

$$\mathcal{W}(G_{i+1} \cup D_{i+1} \cup H_{i+1}) \leqslant F_{j+1}.$$

(2) Case $i+1 = k_{j+1}$ (so now anti-$k_{i+1} = j+1$). We performed update with respect to a differential polynomial p that is a result of reduction of some $\tilde{p} \in G_i \cup D_i$ with respect to A_i. Then

$$\mathcal{W}(A_{i+1}) \leqslant \max(\mathcal{W}(A_i), w(p)) \leqslant F_{j+1}.$$

The third inductive hypothesis is satisfied since $\operatorname{lead} p = s_{j+1}$. Moreover, for every $g \in \bigcup_{t=0}^{i} A_t$ with $s_{j_0} \in \Theta \operatorname{lead}(g)$ for some $j_0 \leqslant j$,

$$w(\operatorname{lcd}(\operatorname{lead}(g), \operatorname{lead}(p))) \leqslant w\left(\operatorname{lcd}(s_{j_0}, s_{j+1})\right) \leqslant F_{j+2}. \tag{6}$$

Since $D_{i+1} \setminus D_i$ consists of some of these polynomials, $G_{i+1} \setminus G_i \subseteq A_i$, and $H_{i+1} \setminus H_i = \{\operatorname{sep}(p), \operatorname{init}(p)\}$, we have

$$\mathcal{W}(G_{i+1} \cup D_{i+1} \cup H_{i+1}) \leqslant \max(\mathcal{W}(G_i \cup D_i \cup H_i), F_{j+2}) = F_{j+2}.$$

Since $w(s_j) \leqslant \mathcal{W}(A_{k_j}) \leqslant F_j$, this completes the proof of Lemma 25. □

Corollary 26. *Let $\{s_i\}$ be an antichain sequence satisfying (5). Then for all $j \geqslant 1$, $w(s_j) \leqslant hf_j$, where $\{f_j\}$ is the Fibonacci sequence.*

Proof. We will prove the statement by induction on j. For $j = 1, 2$, the right-hand side of the bound in Lemma 25 is equal to $h \leqslant hf_j$. Consider $j > 2$. In this case,

$$w(s_j) \leqslant \max\left(h, \max_{l<k<j} w\left(\operatorname{lcd}(s_l, s_k)\right)\right).$$

We have

$$w\left(\operatorname{lcd}(s_l, s_k)\right) \leqslant w(s_l) + w(s_k) \leqslant hf_{j-1} + hf_{j-2} = hf_j.$$

Hence, $w(s_j) \leqslant \max(h, hf_j) = hf_j$. □

Let $\mathfrak{n} = \{1, \ldots, n\}$. Define the *degree* of an element $((i_1, \ldots, i_m), k) \in \mathbb{Z}^m_{\geqslant 0} \times \mathfrak{n}$ to be $i_1 + \ldots + i_m$. Given a weight $w\left(\partial_1^{i_1} \cdots \partial_m^{i_m}\right) = c_1 i_1 + \ldots + c_m i_m$ on Θ, define a map from the set of derivatives ΘY to the set $\mathbb{Z}^m_{\geqslant 0} \times \mathfrak{n}$ by

$$\partial_1^{i_1} \cdots \partial_m^{i_m} y_k \mapsto ((c_1 i_1, \ldots, c_m i_m), k).$$

Note the degree of the image of θy in $\mathbb{Z}_{\geqslant 0} \times \mathfrak{n}$ is equal to the weight of θy in ΘY.

Under this map, the partial order $\preccurlyeq$ on ΘY determines a partial order $\preccurlyeq$ on $\mathbb{Z}^m_{\geqslant 0} \times \mathfrak{n}$ by saying

$$((i_1, \ldots, i_m), k) \preccurlyeq ((j_1, \ldots, j_m), l) \iff k = l \text{ and } i_r \leqslant j_r \text{ for all } r,\ 1 \leqslant r \leqslant m.$$

Thus, every antichain sequence of ΘY determines an antichain sequence of $\mathbb{Z}^m_{\geqslant 0} \times \mathfrak{n}$. Every antichain sequence of $\mathbb{Z}^m_{\geqslant 0} \times \mathfrak{n}$ (and thus of ΘY) is finite (Pierce, 2014, Lemma 4.4).

Given an increasing function $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{\geqslant 0}$, we say that *$f$ bounds the degree growth* of an antichain sequence $S = \{s_1, \ldots, s_k\} \subseteq \mathbb{Z}^m_{\geqslant 0} \times \mathfrak{n}$ if $\deg(s_i) \leqslant f(i)$ for all $1 \leqslant i \leqslant k$. By (Pierce, 2014, Lemma 4.9), there is an upper bound on the length of an antichain sequence of $\mathbb{Z}^m_{\geqslant 0} \times \mathfrak{n}$ with degree growth bounded by f, and this bound depends only on m, n, and f. Let $\mathfrak{L}^n_{f,m}$ be the maximal length of an antichain sequence of $\mathbb{Z}^m_{\geqslant 0} \times \mathfrak{n}$ with degree growth bounded by f.

Theorem 27. *Let $F, K \subseteq \boldsymbol{k}\{Y\}$ be finite subsets with $h = \mathcal{W}(F \cup K)$, $L = \mathfrak{L}^n_{f,m}$, and $\mathcal{A} =$ $\mathsf{Rosenfeld\text{-}Gröbner}(F, K)$, where $f(i) = hf_i$. Then*

$$\mathcal{W}\left(\bigcup_{(A,H)\in\mathcal{A}} (A \cup H)\right) \leqslant hf_{L+1}.$$

Proof. Since $w(\text{pd-red}(p, B)) \leqslant w(p)$ for any $p \in \mathbf{k}\{Y\}$ and weak d-triangular set B, we have $\mathcal{W}(B \cup K) \leqslant \mathcal{W}(A \cup H)$, where $\{(B, K)\} = \mathsf{auto\text{-}partial\text{-}reduce}(A, H)$. Hence, it suffices to bound $\mathcal{W}(G \cup D \cup A \cup H)$ whenever the tuple (G, D, A, H) is added to $\mathcal{S}$ in $\mathsf{Rosenfeld\text{-}Gröbner}$.

By Corollary 26 and the correspondence between antichain sequences of ΘY and $\mathbb{Z}^m_{\geqslant 0} \times \mathfrak{n}$, we obtain an antichain sequence of $\mathbb{Z}^m_{\geqslant 0} \times \mathfrak{n}$ of degree growth bounded by hf_i, so the length of this sequence (and thus the sequence from Lemma 25) is at most L.

In the proof of Lemma 25, it is shown that for all $i \leqslant N$, for $j :=$ anti-k_i, we have

$$\mathcal{W}\left(\bigcup_{t=1}^{i}(G_t \cup D_t \cup A_t \cup H_t)\right) \leqslant F_{j+1}.$$

Since the largest possible j is the length of the antichain sequence (and this j is equal to anti-k_N), for every (G_i, D_i, A_i, H_i), we have

$$\mathcal{W}(G_i \cup D_i \cup A_i \cup H_i) \leqslant F_{L+1}.$$

We bound F_{L+1} in the same way as in Corollary 26:

$$F_{L+1} \leqslant \max\left(h, \max_{l<k<L+1} w\,(\mathrm{lcd}(s_l, s_k))\right) \leqslant hf_{L-1} + hf_L = hf_{L+1}.$$

Since every (G, D, A, H) added to $\mathcal{S}$ is equal to (G_i, D_i, A_i, H_i) for some i,

$$\mathcal{W}\left(\bigcup_{(A,H)\in\mathcal{A}} (A \cup H)\right) \leqslant \max\left(h, \max_{l<k<L+1} w\,(\mathrm{lcd}(s_l, s_k))\right) \leqslant hf_{L+1}. \tag{7}$$

□

We can use Theorem 27 to bound the orders of the output $\mathsf{Rosenfeld\text{-}Gröbner}$. Let $F, K \subseteq \mathbf{k}\{Y\}$ be two finite subsets, and define a weight w on Θ such that

$$\mathcal{W}(F \cup K) = \mathcal{H}(F \cup K). \tag{8}$$

This can always be done by letting $w(\theta) = \mathrm{ord}(\theta)$ for all derivatives θ, but there are sometimes other weights that lead to equation (8) being satisfied.

Example 28. *We provide examples of differential polynomials f that arise as part of systems of PDEs for which it is possible to construct a nontrivial weight w such that $w(f) = \mathrm{ord}(f)$. We note that we are not applying* $\mathsf{Rosenfeld\text{-}Gröbner}$ *to these examples; we simply present them to demonstrate that there are nontrivial weights satisfying equation* (8)*.*

(1) *Consider the heat equation*

$$u_t - \alpha \cdot (u_{xx} + u_{yy}) = 0, \quad f(u) := \partial_t u - \alpha \cdot (\partial_x^2 u + \partial_y^2 u) \in \boldsymbol{k}\{u\},$$

where $u(x, y, t)$ is the unknown, α is a positive constant, and $\boldsymbol{k}\{u\}$ has derivations $\{\partial_x, \partial_y, \partial_t\}$. If we define a weight w on Θ by

$$w\left(\partial_x^i \partial_y^j \partial_t^k\right) = i + j + 2k,$$

then $w(f) = 2 = \operatorname{ord}(f)$.

(2) *Consider the K-dV equation*

$$\phi_t + \phi_{xxx} + 6\phi\phi_x = 0, \quad f(\phi) := \partial_t\phi + \partial_x^3\phi + 6\phi\partial_x\phi \in \boldsymbol{k}\{\phi\},$$

where $\phi(x, t)$ is the unknown and $\boldsymbol{k}\{\phi\}$ has derivations $\{\partial_x, \partial_t\}$. Define a weight w on Θ by

$$w\left(\partial_x^i \partial_t^j\right) = i + 3j,$$

so that $w(f) = 3 = \operatorname{ord}(f)$.

Using Theorem 27, and (8), we obtain the following order bound for the output of Rosenfeld-Gröbner:

Corollary 29. *Let $F, K \subseteq \boldsymbol{k}\{Y\}$ be finite subsets with $h = \mathcal{H}(F \cup K)$, $L = \mathfrak{L}^n_{f,m}$, $\mathcal{A} =$ Rosenfeld-Gröbner(F, K), where $f(i) = hf_i$ with $\{f_i\}$ the Fibonacci sequence. Let $w\left(\partial_1^{i_1} \cdots \partial_m^{i_m}\right) = c_1 i_1 + \cdots + c_m i_m$ be a weight defined on Θ such that $\mathcal{W}(F \cup K) = \mathcal{H}(F \cup K)$. Then, for all $g \in \mathcal{A}$,*

$$\operatorname{ord}(g, \partial_i) \leqslant \frac{hf_{L+1}}{c_i}$$

5. Refined bound for m = 2

In this section we give a refined weight bound for the Rosenfeld-Gröbner algorithm for the case of two derivations. Let c_1 and c_2 be positive integers and w be the weight defined by

$$w\left(\partial_1^a \partial_2^b\right) = c_1 a + c_2 b.$$

Proposition 30. *Let $h \geqslant 0$ and an antichain $\{s_1, \ldots, s_L\}$ in Θy, where $s_i = \partial_1^{s_{i,1}} \partial_2^{s_{i,2}} y$, $1 \leqslant i \leqslant L$, be such that*

$$w(s_j) \leqslant \max\left(h, \max_{l<k<j} w(\operatorname{lcd}(s_l, s_k))\right). \tag{9}$$

Then either, for every i, $1 \leqslant i \leqslant L$, we have

$$c_1 s_{i,1} \leqslant f_{h+2} - 1 \quad \text{and} \quad c_2 s_{i,2} \leqslant f_{h+3} - 2$$

or, for every i, $1 \leqslant i \leqslant L$, we have

$$c_2 s_{i,2} \leqslant f_{h+2} - 1 \quad \text{and} \quad c_1 s_{i,1} \leqslant f_{h+3} - 2.$$

Proof. We will prove the proposition by induction on h. If $h = 0$, then $L = 1$, $s_{1,1} = s_{1,2} = 0$, so

$$c_1 s_{1,1} = 0 \leqslant f_2 - 1 \quad \text{and} \quad c_2 s_{1,2} = 0 \leqslant f_3 - 2.$$

Now consider $h \geqslant 1$. First note that for all $r > 1$ and all $c > 0$,

$$f_r + c \leqslant f_{r+c}. \tag{10}$$

This can be seen by applying, for all $r > 1$, $f_r + 1 \leqslant f_{r+1}$, c times. Assume that, for every j, $1 \leqslant j \leqslant L$, $s_{j,1} > 0$. In particular, this means that $h \geqslant w(s_1) \geqslant c_1$. The antichain sequence $\{\tilde{s}_1, \dots, \tilde{s}_L\}$, where

$$\tilde{s}_j = \partial_1^{s_{j,1}-1} \partial_2^{s_{j,2}} y, \quad 1 \leqslant j \leqslant L,$$

satisfies the inductive hypothesis for $h - c_1$ in place of h. Thus, we have the following two cases:

(1) For every j, $1 \leqslant j \leqslant L$,

$$c_1 \tilde{s}_{j,1} \leqslant f_{h-c_1+2} - 1 \quad \text{and} \quad c_2 \tilde{s}_{j,2} \leqslant f_{h-c_1+3} - 2.$$

This implies that for every j, $1 \leqslant j \leqslant L$,

$$c_1 s_{j,1} = c_1(\tilde{s}_{j,1} + 1) \leqslant f_{h-c_1+2} + c_1 - 1 \leqslant f_{h+2} - 1$$

(the latter inequality follows from (10)), and

$$c_2 s_{j,2} = c_2 \tilde{s}_{j,2} \leqslant f_{h-c_1+3} - 2 \leqslant f_{h+3} - 2.$$

(2) For every j, $1 \leqslant j \leqslant L$,

$$c_1 \tilde{s}_{j,1} \leqslant f_{h-c_1+3} - 2 \quad \text{and} \quad c_2 \tilde{s}_{j,2} \leqslant f_{h-c_1+2} - 1.$$

This implies that for every j, $1 \leqslant j \leqslant L$,

$$c_1 s_{j,1} = c_1(\tilde{s}_{j,1} + 1) \leqslant f_{h-c_1+3} + c_1 - 2 \leqslant f_{h+3} - 2$$

(by (10)), and

$$c_2 s_{j,2} = c_2 \tilde{s}_{j,2} \leqslant f_{h-c_1+2} - 1 \leqslant f_{h+2} - 1.$$

The same argument works for the case in which $s_{j,2} > 0$ for every j, $1 \leqslant j \leqslant L$.

Hence, without loss of generality, we can assume that there exist j_1, j_2, a, and b such that $1 \leqslant j_1 < j_2 \leqslant L$, $s_{j_1} = \partial_1^a y$, $s_{j_2} = \partial_2^b y$, and there is no $j < j_2$ with $s_{j,1} = 0$. We construct an antichain sequence $\{\tilde{s}_1, \dots, \tilde{s}_{j_2-1}\}$ by the formula

$$\tilde{s}_i = \partial_1^{\tilde{s}_{i,1}} \partial_2^{\tilde{s}_{i,2}} y, \quad \tilde{s}_{i,1} = s_{i,1} - 1, \quad \tilde{s}_{i,2} = s_{i,2};$$

this satisfies the inductive hypothesis for $h - c_1$ in place of h. Again, we have two cases:

(1) For every i, $1 \leqslant i < j_2$, $c_1 \tilde{s}_{i,1} \leqslant f_{h-c_1+2} - 1$ and $c_2 \tilde{s}_{i,2} \leqslant f_{h-c_1+3} - 2$.

(2) For every i, $1 \leqslant i < j_2$, $c_1 \tilde{s}_{i,1} \leqslant f_{h-c_1+3} - 2$ and $c_2 \tilde{s}_{i,2} \leqslant f_{h-c_1+2} - 1$.

In particular,

$$c_1 a \leqslant \max(f_{h-c_1+2} - 1, f_{h-c_1+3} - 2) + c_1 = f_{h-c_1+3} + c_1 - 2 \leqslant f_{h+2} - 1,$$

the latter inequality is due to (10). In both cases, due to (10),

$$\max_{l<k<j_2} w(\operatorname{lcd}(s_l, s_k)) \leqslant f_{h-c_1+2} - 1 + f_{h-c_1+3} + c_1 - 2 = f_{h-c_1+4} + c_1 - 3 \leqslant f_{h+3} - 2.$$

Due to (10), $f_{h+3} - 2 \geqslant h + f_3 - 2 = h$. Hence

$$c_2 b \leqslant \max\left(h, \max_{l<k<j_2} w(\operatorname{lcd}(s_l, s_k))\right) \leqslant f_{h+3} - 2.$$

Since, for all i, $1 \leqslant i \leqslant L$,

$$c_1 s_{i,1} \leqslant c_1 a \quad \text{and} \quad c_2 s_{i,2} \leqslant c_2 b$$

by the definition of an antichain sequence, the proof is complete. □

Similarly to Theorem 27, we can prove the following.

Corollary 31. *Let $F, K \subseteq \mathbf{k}\{y\}$ (with respect to two derivations) be finite subsets with $h = \mathcal{W}(F \cup K)$ and $\mathcal{A} = \mathsf{Rosenfeld\text{-}Gröbner}(F, K)$. Then*

$$\mathcal{W}\left(\bigcup_{(A,H)\in\mathcal{A}} (A \cup H)\right) \leqslant f_{h+4} - 3.$$

Proof. We will use inequality (7) from the proof of Theorem 27. In particular, it states that

$$\mathcal{W}\left(\bigcup_{(A,H)\in\mathcal{A}} (A \cup H)\right) \leqslant \max\left(h, \max_{l<k<L+1} w(\operatorname{lcd}(s_l, s_k))\right),$$

where L is the length of the corresponding antichain $\{s_1, \ldots, s_L\}$. By Proposition 30,

$$\max\left(h, \max_{l<k<L+1} w(\operatorname{lcd}(s_l, s_k))\right) \leqslant f_{h+2} - 1 + f_{h+3} - 2 = f_{h+4} - 3.$$ □

Proposition 30 also allows us to significantly improve the bound from (Gustavson et al., 2016b) for $m = 2$ and arbitrary n. We define the number $b(n, h)$ recursively by

$$b(1,h) = h+1, \quad b(2,h) = f_{h+4} - 1 + h, \quad b(n,h) = h f_{b(n-1,h)+1} + b(n-1,h) + 1 \text{ for } n > 2. \tag{11}$$

The proof of the following proposition is similar to the proof of (Freitag and León Sánchez, 2016, Lemma 3.8).

Proposition 32. *For every antichain sequence $\{s_1, \ldots, s_L\}$ in ΘY (with respect to two derivations), where $Y = \{y_1, \ldots, y_n\}$, satisfying*

$$w(s_j) \leqslant \max\left(h, \max_{l<k<j} w(\operatorname{lcd}(s_l, s_k))\right), \tag{12}$$

we have $L \leqslant b(n, h)$.

Proof. We will prove the proposition by induction on n. For $n = 1$, it follows from (Freitag and León Sánchez, 2016, Lemma 3.7).

Assume that $n > 1$. For every i, $1 \leqslant i \leqslant n$, let

$$g_i := \min\{j \mid s_j \in \Theta y_i\}.$$

Reordering the y's if necessary, we can assume that $g_1 < g_2 < \ldots < g_n$.

First, we separately consider the case $n = 2$. By the definition of g_2, the sequence $\{s_1, \ldots, s_{g_2-1}\}$ is an antichain sequence in Θy_1. Proposition 30 implies that, for every $i < j < g_2$,

$$w(\operatorname{lcd}(s_i, s_j)) \leqslant f_{h+4} - 3,$$

so $w(s_{g_2}) \leqslant f_{h+4} - 3$. Hence, due to (Freitag and León Sánchez, 2016, Lemma 3.7), since $\operatorname{ord}(s) \leqslant w(s)$,

$$\Theta y_2 \cap \{s_1, \ldots, s_L\} \leqslant f_{h+4} - 2.$$

Thus, in total, there are at most $h + 1 + f_{h+4} - 2 = b(2, h)$ of them.

Now let $n > 2$. For every l, $3 \leqslant l \leqslant n$, the antichain sequence $\{s_1, \ldots, s_{g_l-1}\}$ satisfies the inductive hypothesis. Hence, its length, $g_l - 1$, does not exceed $b(l-1, h)$, so $g_l \leqslant b(l-1, h) + 1$. Thus, due to Corollary 26, for every l, $3 \leqslant l \leqslant n$, there exists an element of $\{s_1, \ldots, s_L\} \cap \Theta y_l$ of weight at most $hf_{g_l} \leqslant hf_{b(l-1,h)+1}$. Thus, the length of the antichain sequence can be bounded using (Freitag and León Sánchez, 2016, Lemma 3.7) as

$$L \leqslant (h+1) + (f_{h+4} - 2) + \sum_{l=3}^{n} (hf_{b(l-1,h)+1} + 1) = b(n, h). \qquad \square$$

Corollary 33. *Let $F, K \subseteq \boldsymbol{k}\{Y\}$ (with respect to two derivations) be finite subsets with $h = \mathcal{W}(F \cup K)$ and $\mathcal{A} = \mathsf{Rosenfeld\text{-}Gröbner}(F, K)$. Then*

$$\mathcal{W}\left(\bigcup_{(A,H)\in\mathcal{A}} (A \cup H)\right) \leqslant hf_{b(n,h)+1}.$$

Proof. The proof is the same as for Theorem 27 with $b(n, h)$ from Proposition 32 as a bound for the length of the antichain sequence. $\square$

6. Specific values

6.1. $m = 2$

When $m = 2$, we can use the results of Corollaries 31 and 33 to bound the weight of the output of the Rosenfeld-Gröbner algorithm.

If $\mathcal{W}(F \cup K) = \mathcal{H}(F \cup K) = h$, we can use Corollary 29 to produce perhaps sharper bounds for the order of the elements of $\mathsf{Rosenfeld\text{-}Gröbner}(F, K)$ with respect to particular derivations. In the examples that follow, we calculate upper bounds for $\operatorname{ord}(g, \partial_1)$ for $g \in \mathsf{Rosenfeld\text{-}Gröbner}(F, K)$, where $w\left(\partial_1^{i_1} \cdots \partial_m^{i_m}\right) = c_1 i_1 + \ldots + c_m i_m$ with c_1 equal to either 2 or 3. We note that in the tables that follow, "N/A" appears whenever we cannot have the given initial order h with given c_i as part of the weight function.

(1) Assume $m = 2$ and $n = 1$. Then by Corollary 31, the weights of the output polynomials are bounded by $f_{h+4} - 3$, which results in the following table:

h	1	2	3	4	5	6	7	8	9	10
$f_{h+4} - 3$	2	4	10	18	31	52	86	141	230	374
$\operatorname{ord}(g, \partial_1)$, $c_1 = 2$	N/A	2	5	9	15	26	43	70	115	187
$\operatorname{ord}(g, \partial_1)$, $c_1 = 3$	N/A	N/A	3	6	10	17	28	47	76	124

(2) Assume that $m = 2$ and n is arbitrary. Then by Corollary 33, the weights of the output polynomials are bounded by $hf_{b(n,h)+1}$, with $b(n, h)$ defined by (11). This results in the following table:

n	h	$b(n, h)$	$hf_{b(n,h)+1}$	$\operatorname{ord}(g, \partial_1)$, $c_1 = 2$	$\operatorname{ord}(g, \partial_1)$, $c_1 = 3$
2	1	5	8	N/A	N/A
2	2	9	110	55	N/A
2	3	15	2,961	1,480	987
2	4	24	300, 100	150, 050	100, 033
2	5	38	316, 229, 930	158, 114, 965	105, 409, 976
3	1	14	610	N/A	N/A

(3) Below is a table comparing the new weight upper bound $hf_{b(n,h)+1}$ for $m = 2$ (via Corollary 33) to the previous order upper bound $f(\mathfrak{L}^n_{f,2} + 1)$, where $f(i) = hf_i$, given in (Gustavson et al., 2016b) for some specific values of n and h. As discussed in Corollary 29, given a specific weight w such that $\mathcal{W}(F \cup K) = \mathcal{H}(F \cup K)$ and a weight upper bound, we can produce order upper bounds with respect to each derivation that are no greater than the weight upper bound.

n	h	Old: $f(\mathfrak{L}^n_{f,2} + 1)$	New: $hf_{b(n,h)+1}$
1	3	15	10
1	6	126	52
1	9	801	141
1	12	4,524	984
1	15	23,955	4,178
2	1	8	8
2	2	178	110
2	3	32,838	2,961
2	4	252, 983, 944	300,100
2	5	$\leqslant 10^{16}$	316, 229, 930
3	1	610	610
3	2	$\leqslant 10^{40}$	$\leqslant 10^{26}$

6.2. $m > 2$

For an arbitrary number of derivations ($m > 2$), by Theorem 27, the weight of the output polynomials of the Rosenfeld-Gröbner algorithm is bounded above by $hf_{L+1} = f(L + 1)$, where $f(i) = hf_i$ and $L = \mathfrak{L}^n_{f,m}$. In order to apply this result, we need to be able to effectively compute

$\mathfrak{L}^n_{f,m}$. (Pierce, 2014) only proved the existence of this number, without an analysis of how to construct it. (Freitag and León Sánchez, 2016) constructed an upper bound for $m = 1, 2$. The first analysis for the case of arbitrary m appears in (León Sánchez and Ovchinnikov, 2016).

Let $f\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{\geqslant 0}$ be an increasing function. Let us define a function $\Psi_{f,m}\colon \mathbb{Z}_{>0} \times \mathbb{Z}^m_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0}$ by the following relations:

$$\begin{cases} \Psi_{f,m}(i, (0, \ldots, 0, u_m)) = i, \\ \Psi_{f,m}(i-1, (u_1, \ldots, u_r, 0, \ldots, 0, u_m)) \\ \qquad = \Psi_{f,m}(i, (u_1, \ldots, u_r - 1, f(i) - f(i-1) + u_m + 1, 0, \ldots, 0)), & r < m-1, u_r > 0, \\ \Psi_{f,m}(i-1, (u_1, \ldots, u_m)) \\ \qquad = \Psi_{f,m}(i, (u_1, \ldots, u_{m-1} - 1, f(i) - f(i-1) + u_m + 1)), & u_{m-1} > 0. \end{cases}$$

Proposition 34 ((León Sánchez and Ovchinnikov, 2016, Corollary 3.10)). *The maximal length of an antichain sequence in $\mathbb{Z}^m_{\geqslant 0}$ with degree growth bounded by f does not exceed*

$$\Psi_{f,m}(1, (f(1), 0, \ldots, 0)).$$

Let us also define the sequence $\psi_0, \psi_1, \ldots$ by the relations $\psi_0 = 0$ and

$$\psi_{i+1} = \Psi_{f_i,m}(1, (f_i(1), 0, \ldots, 0)) + \psi_i, \quad f_i(x) := f(x + \psi_i).$$

Proposition 35 ((León Sánchez and Ovchinnikov, 2016, Corollary 3.14)). *The maximal length of an antichain sequence in $\mathbb{Z}^m_{\geqslant 0} \times \mathfrak{n}$ with degree growth bounded by f does not exceed ψ_n.*

Now, let us apply this technique to the function $f(i) = hf_i$. Then, by Theorem 27, an upper bound on the weights of the output of Rosenfeld-Gröbner will be $f(\mathfrak{L}^n_{f_2,m} + 1)$. In general, we do not have a formula for $\mathfrak{L}^n_{f,m}$ for arbitrary h, m, n that improves the one given in Proposition 35; however, we can compute $\mathfrak{L}^n_{f,m}$ for some specific values of h, m, n.

(1) Assume that $m = 3$ and $n = 1$. We can construct the maximal length antichain sequence of $\mathbb{Z}^3_{\geqslant 0}$ using the methods of (León Sánchez and Ovchinnikov, 2016) and the function $f(i) = hf_i$, resulting in the following sequence:

$$\begin{aligned} &(h,0,0), (h-1,1,0), (h-1,0,h+1), (h-2,2h+2,0), \ldots, \\ &(h-2,0,hf_{2h+6}-(h-2)), \ldots, (h-i, hf_{c_{i-1}+1}-(h-i), 0), \ldots, \\ &\qquad\qquad (h-i, 0, hf_{c_i}-(h-i)), \ldots, (0, hf_{c_{h-1}+1}, 0), \ldots, (0,0,hf_{c_h}), \end{aligned}$$

where the sequence c_i is given by $c_0 = 1$ and for $1 \leqslant i \leqslant h$,

$$c_i = c_{i-1} + 1 + hf_{c_{i-1}+1} - (h-i).$$

As a result, we see that the maximal length of an antichain sequence is equal to c_h.

(2) Below is a table of some maximal lengths $\mathfrak{L}^n_{f,m}$ and weights $f(\mathfrak{L}^n_{f,m} + 1)$, where $f(i) = hf_i$, for $m = 3$, 4, and 5:

m	n	h	length	weight
3	1	1	3	3
3	1	2	10	178
4	1	1	5	8
5	1	1	20	10,946

7. Order lower bound

This section gives a lower bound for the orders of the output of Rosenfeld-Gröbner, coming from the lower bound for degrees of elements of a Gröbner basis from (Yap, 1991). To be specific, we show that for m, h sufficiently large, there is a set of r differential polynomials $F \subseteq \mathbf{k}\{y\}$ of order at most h, where $\mathbf{k}$ is equipped with m derivations, $r \sim m/2$, and $\mathbf{k}$ is constant with respect to all of the derivations, such that if $\mathcal{A} = \text{Rosenfeld-Gröbner}(F, \{1\})$, then

$$\mathcal{H}\left(\bigcup_{(A,H)\in\mathcal{A}} (A \cup H)\right) \geqslant h^{2^r}. \tag{13}$$

The arguments presented here are standard, and we include them for completeness. We first note the following standard fact about differential ideals generated by linear differential polynomials.

Proposition 36. *Suppose $F, K \subseteq \boldsymbol{k}\{Y\}$ are composed of linear differential polynomials. Then the output of* Rosenfeld-Gröbner(F, K) *is either empty or consists of a single regular differential system (A, H) with A and H both composed of linear differential polynomials.*

Suppose now we apply Rosenfeld-Gröbner to $(F, \{1\})$, where F consists of linear differential polynomials, in order to obtain a regular decomposition of $\{F\}$. Since every element of F is linear, $[F]$ is a prime differential ideal, so by Proposition 36, we have

$$[F] = \{F\} = [A] : H^\infty$$

for some regular differential system (A, H), with A and H both composed of linear differential polynomials. Since every element of A is linear, after performing scalar multiplications and addition, A can be transformed to an autoreduced set $\bar{A}$ without affecting the leaders and orders of elements of A. Since (A, H) is a regular differential system, $\bar{A}$ is a characteristic set of $[F]$. So, it suffices to find a lower bound on the orders of elements of linear characteristic sets in $\mathbf{k}\{Y\}$.

There is a well-studied one-to-one correspondence between polynomials in $\mathbf{k}[x_1, \ldots, x_m]$ and homogeneous linear differential polynomials in $\mathbf{k}\{y\}$ with m derivations and $\mathbf{k}$ a field of constants:

$$\sum c_{i_1,\ldots,i_m} x_1^{i_1} \cdots x_m^{i_m} \leftrightarrow \sum c_{i_1,\ldots,i_m} \partial_1^{i_1} \cdots \partial_m^{i_m} y. \tag{14}$$

Any orderly ranking on Θy then determines a graded monomial order on $\mathbf{k}[x_1, \ldots, x_m]$.

Given a polynomial $f \in \mathbf{k}[x_1, \ldots, x_m]$, let $\tilde{f} \in \mathbf{k}\{y\}$ be its corresponding differential polynomial under (14). By the discussion above, if we have a collection of polynomials $f_1, \ldots, f_r \in \mathbf{k}[x_1, \ldots, x_m]$, we can construct a characteristic set $C = \{C_1, \ldots, C_s\}$ of $[\tilde{f}_1, \ldots, \tilde{f}_r] \subseteq \mathbf{k}\{y\}$ consisting of homogeneous linear differential polynomials, and so each $C_i \in \mathbf{k}\{y\}$ is in fact equal to $\tilde{g}_i$ for some $g_i \in \mathbf{k}[x_1, \ldots, x_m]$.

Proposition 37 (cf. (Wu, 2005, page 352),(Gerdt, 2005))**.** *With the notation above, $\{g_1, \ldots, g_s\} \subseteq \boldsymbol{k}[x_1, \ldots, x_m]$ is a Gröbner basis of the ideal $I = (f_1, \ldots, f_r)$.*

By Proposition 37, we can thus find a lower bound for the orders of the output of Rosenfeld-Gröbner via a lower bound for the degrees of elements of a Gröbner basis, as we do in the following example.

Example 38. *This example demonstrates the lower bound* (13) *for the orders of the output of* Rosenfeld-Gröbner. *In (Yap, 1991, Section 8), for m, h sufficiently large, a collection of m algebraic polynomials $f_1, \ldots, f_r$ of degree at most h in m algebraic indeterminates, with $r \sim m/2$, is constructed such that any Gröbner basis of $(f_1, \ldots, f_r)$ with respect to a graded monomial order has an element of degree at least h^{2^r}.*

As a result of the previous discussion, we have a collection of differential polynomials $F = \tilde{f}_1, \ldots, \tilde{f}_r \in \boldsymbol{k}\{y\}$ of order h with m derivations such that any linear characteristic set of $[\tilde{f}_1, \ldots, \tilde{f}_r]$ will contain a differential polynomial of order at least h^{2^r}. Since in this case $\{(A, H)\} =$ Rosenfeld-Gröbner$(F, \{1\})$ *can be transformed into a linear characteristic set without affecting the orders of the elements, this means that*

$$\mathcal{H}(A \cup H) \geqslant h^{2^r}.$$

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