# Singular Values of Tensors

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# Tensor Rank

$\mathbb{F}$ a field, $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$
$V^{(i)} \cong \mathbb{F}^{n_i}$ for $i = 1, 2, \ldots, d$
$V = V^{(1)} \otimes V^{(2)} \otimes \cdots \otimes V^{(d)} \cong \mathbb{F}^{n_1 \times \cdots \times n_d}$ tensor product space

**Definition**

A *simple* tensor is a tensor of the form $v^{(1)} \otimes v^{(2)} \otimes \cdots \otimes v^{(d)}$ $(v^{(i)} \in V^{(i)})$.

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## Definition (tensor rank)

The *rank* of $T$ is the smallest positive integer $r$ such that $T$ is the sum of $r$ simple tensors.

# Matrix Multiplication Tensor

$$M_d = \sum_{i=1}^{d} \sum_{j=1}^{d} \sum_{k=1}^{d} e_{i,j} \otimes e_{j,k} \otimes e_{k,i} \in \mathbb{F}^{n \times n} \otimes \mathbb{F}^{n \times n} \otimes \mathbb{F}^{n \times n}$$

clearly $\operatorname{rank}(M_d) \leq d^3$

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**Theorem (Strassen)**

*if* $\operatorname{rank}(M_d) = s$ *then complexity of* $n \times n$ *matrix multiplication is* $O(n^{\log_d(s)})$ *(the standard algorithm is* $O(n^3)$*)*

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**Theorem (Strassen)**

$\operatorname{rank}(M_2) \leq 7$, *so complexity of* $n \times n$ *matrix multiplication is* $O(n^{\log_2(7)}) = O(n^{2.81})$

Current record: $O(n^{2.373})$ (Le Gall)

# The Canonical Polyadic (CP) Model

aka PARAFAC, CANDECOMP

**Problem**

*Given a tensor* $T \in V = V^{(1)} \otimes \cdots \otimes V^{(d)}$, *write* $T = v_1 + v_2 + \cdots + v_r$ *where* $v_1, v_2, \ldots, v_r$ *are simple tensors and* $r$ *is minimal.*

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- the CP-decomposition is sometimes unique, not always
- not numerically stable: there is no upper bound for $\max\{\|v_1\|, \ldots, \|v_r\|\}$ as a function of $\|T\|$.
- difficult to compute: for many tensors of interest the rank is unknown

Many numerical applications in psychometrics, signal processing, numerical linear algebra, computer vision, numerical analysis, data mining, graph analysis, neuroscience, chemometrics, etc.

# The Canonical Polyadic (CP) Model

To make it more numerically stable, we could allow an error term.

**Problem**

*Given a tensor $T \in V$ and fixed $r$, write $T = v_1 + v_2 + \cdots + v_r + E$ where $v_1, v_2, \ldots, v_r$ are simple tensors and $\|E\|$ is minimal.*

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This optimization problem might be ill-posed. Define

$$T = e_2 \otimes e_1 \otimes e_1 + e_1 \otimes e_2 \otimes e_1 + e_1 \otimes e_1 \otimes e_2,$$

$$v_1(t) = t^{-1}(e_1 + te_2) \otimes (e_1 + te_2) \otimes (e_1 + te_2), \quad v_2(t) = -t^{-1} e_1 \otimes e_1 \otimes e_1$$

Then $T = v_1(t) + v_2(t) + E(t)$ with $\|E(t)\| = t\sqrt{3+t}$.

So we can write $T = v_1 + v_2 + E$ with $\|E\| = \varepsilon$ for every $\varepsilon > 0$, but we cannot write $T = v_1 + v_2 + E$ with $\|E\| = 0$.

# Motivation: Compressed Sensing and Convex Relaxation

For $x \in \mathbb{R}^n$, its sparsity is measured by $\|x\|_0 = \#\{i \mid x_i \neq 0\}$.

**Problem**

*Given $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, find a solution $x \in \mathbb{R}^n$ for $Ax = b$ with $\|x\|_0$ minimal (a sparsest solution).*

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But, $\|\cdot\|_0$ is not convex and this optimization problem is difficult, so instead we consider:

**Problem (Basis Pursuit)**

*Given $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, find a solution $x \in \mathbb{R}^n$ for $Ax = b$ with $\|x\|_1$ minimal.*

Basis Pursuit can be solved by linear programming and is generally fast. Under reasonable assumptions, Basis Pursuit also gives the sparsest solutions (Candès-Tao, Donoho).

# The Nuclear and Spectral Norms

Low rank tensors are sparse in some sense, and by convex relaxation:

**Definition**

The nuclear norm $\|T\|_\star$ is the smallest value of $\sum_{i=1}^r \|v_i\|$ where $T = \sum_{i=1}^r v_i$ and $v_1, \dots, v_r$ are simple tensors. (well-defined)

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$V^{(1)}, \dots, V^{(d)}, V$ a spaces with a positive definite bilinear/hermitian form $\langle \cdot, \cdot \rangle$

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**Definition**

The spectral norm is defined by

$$\|T\|_\sigma = \max\{|\langle T, v\rangle| \mid v \text{ simple tensor with } \|v\| = 1\}.$$

The spectral norm is *dual* to the nuclear norm, in particular

$$|\langle T, S\rangle| \le \|T\|_\star \|S\|_\sigma \quad \text{for all tensors } S, T.$$

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If $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$ are the singular values of $A$, then $\|A\|_\star = \lambda_1 + \cdots + \lambda_r$, $\|A\|_\sigma = \lambda_1$ (spectral/operator norm) and $\|A\| = \|A\|_2 = \|A\|_F = \sqrt{\lambda_1^2 + \cdots + \lambda_r^2}$ (Euclidean/Frobenius norm)

# Example: Determinant Tensor

$$D_n = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) e_{\sigma(1)} \otimes e_{\sigma(2)} \otimes \cdots \otimes e_{\sigma(n)} \in \mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n.$$

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$$\|D_n\|_\sigma = \max\{|\det(v_1 v_2 \cdots v_n)| \mid \|v_1\| = \cdots = \|v_n\| = 1\} = 1$$

$$\|D_n\|_\star = \|D_n\|_\star \|D_n\|_\sigma \geq \langle D_n, D_n \rangle = n!.$$

so $\|D_n\|_\star \geq n!$

**Theorem (D.)**

$\|D_n\|_\star = n!$

# Example: Permanent Tensor

$$P_n = \sum_{\sigma \in S_n} e_{\sigma(1)} \otimes e_{\sigma(2)} \otimes \cdots \otimes e_{\sigma(n)} \in \mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n.$$

$$\|P_n\|_\sigma = \max\{|\operatorname{perm}(v_1 v_2 \cdots v_n)| \mid \|v_1\| = \cdots = \|v_n\| = 1\}$$

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**Theorem (Carlen, Lieb and Moss, 2006)**

$$\max\{\operatorname{perm}(v_1 v_2 \cdots v_n) \mid \|v_1\| = \cdots = \|v_n\| = 1\} = n!/n^{n/2}$$

$$\frac{n!}{n^{n/2}}\|P_n\|_\star = \|P_n\|_\star \|P_n\|_\sigma \geq \langle P_n, P_n \rangle = n!.$$

so $\|P_n\|_\star \geq n^{n/2}$

# Example: Permanent Tensor

**Theorem (Glynn 2010)**

$$P_n = \frac{1}{2^{n-1}} \sum_{\delta} \left( \prod_{i=1}^n \delta_i \right) \left(\sum_{i=1}^n \delta_i e_i\right) \otimes \cdots \otimes \left(\sum_{i=1}^n \delta_i e_i\right)$$

*where $\delta$ runs over $\{1\} \times \{-1, 1\}^{n-1}$.*

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*where* $\delta$ *runs over* $\{1\} \times \{-1, 1\}^{n-1}$.

In particular, $\binom{n}{\lfloor n/2 \rfloor} \leq \text{rank}(P_n) \leq 2^{n-1}$ and $\|P_n\|_\star \leq n^{n/2}$, so

**Theorem (D.)**

$$\|P_n\|_\star = n^{n/2}$$

# The Convex Decomposition (CoDe) Model

**Problem**

*Given a tensor $T \in V$, write $T = v_1 + v_2 + \cdots + v_r$ where $v_1, v_2, \ldots, v_r$ are simple tensors and $\|v_1\| + \|v_2\| + \cdots + \|v_r\|$ is minimal.*

Well-defined but the decomposition may not be unique.

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Well-defined but the decomposition may not be unique.

We can also allow an error term ...

# The Convex Decomposition (CoDe) Model

**Problem**

*Given a tensor $T \in V$ and fixed $x \geq 0$, write $T = v_1 + v_2 + \cdots + v_r + E$ where $r$ is a nonnegative integer, $v_1, v_2, \ldots, v_r$ are simple tensors, $\|v_1\| + \|v_2\| + \cdots + \|v_r\| \leq x$ and $\|E\|$ is minimal.*

The decomposition is not unique.

If $A = v_1 + v_2 + \cdots + v_r$ then $A$ is the unique tensor with $\|A\|_\star \leq x$ for which $\|T - A\|$ is minimal.

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If $A = v_1 + v_2 + \cdots + v_r$ then $A$ is the unique tensor with $\|A\|_\star \leq x$ for which $\|T - A\|$ is minimal.

**Problem**

*Given a tensor* $T \in V$ *and fixed* $y \geq 0$, *write* $T = v_1 + v_2 + \cdots + v_r + E$ *where* $r$ *is a nonnegative integer,* $v_1, v_2, \ldots, v_r$ *are simple tensors,* $\|E\| \leq y$ *and* $\|v_1\| + \|v_2\| + \cdots + \|v_r\|$ *is minimal.*

# $t$-Orthogonality

## Definition

Simple unit tensors $v_1, v_2, \ldots, v_r$ are $t$-orthogonal if

$$\sum_{i=1}^{r} |\langle v_i, w\rangle|^{2/t} \leq 1$$

for every simple tensor $w$ with $\|w\|_2 = 1$.

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**Theorem (D.)**

*If $v_1, \ldots, v_r \in V$ are $t$-orthogonal, then $r \leq \dim(V)^{1/t}$.*

# Horizontal and Vertical Tensor Product

**Theorem ("horizontal tensor product", D.)**

*If $v_1, \dots, v_r$ are $t$-orthogonal, and $w_1, \dots, w_r$ are $s$-orthogonal, then $v_1 \otimes w_1, \dots, v_r \otimes w_r$ are $(s+t)$-orthogonal.*

# Horizontal and Vertical Tensor Product

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If $V = V^{(1)} \otimes \cdots \otimes V^{(d)}$ and $W = W^{(1)} \otimes \cdots \otimes W^{(d)}$, then

$$V \boxtimes W := (V^{(1)} \otimes W^{(1)}) \otimes \cdots \otimes (V^{(d)} \otimes W^{(d)}).$$

$$(v_1 \otimes \cdots \otimes v_d) \boxtimes (w_1 \otimes \cdots \otimes w_d) = (v_1 \otimes w_1) \otimes (v_2 \otimes w_2) \otimes \cdots \otimes (v_d \otimes w_d)$$

**Theorem ("vertical tensor product", D.)**

*If $v_1, v_2, \dots, v_r \in V$ and $w_1, \dots, w_s \in W$ are $t$-orthogonal, then $\{v_i \boxtimes w_j \mid 1 \leq i \leq r, 1 \leq j \leq s\}$ are $t$-orthogonal.*

# The Diagonal Singular Value Decomposition

**Definition**

Suppose that $(\star) : T = \sum_{i=1}^{r} \lambda_i v_i$ such that $\lambda_1 \geq \cdots \geq \lambda_r > 0$ and $v_1, \ldots, v_r$ are 2-orthogonal simple tensors of unit length, then $(\star)$ is called a *diagonal singular value decomposition* of $T$ (DSVD).

If $d = 2$ (tensor product of 2 spaces) then the DSVD is the usual singular value decomposition. For $d > 2$, the DSVD is different from the *Higher Order Singular Value Decomposition* defined by De Lathauer, De Moor, and Vandewalle. Not every tensor has a DSVD.

# The Diagonal Singular Value Decomposition

**Theorem (D.)**

*If T has a DSVD then*

$$\|T\|_\star = \sum_i \lambda_i, \quad \|T\|_2 = \sqrt{\sum_i \lambda_i^2}, \quad \|T\|_\sigma = \lambda_1$$

# The Diagonal Singular Value Decomposition

## Theorem (D.)

*If $T$ has a DSVD then*

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## Theorem (D.)

*If $\lambda_1 > \lambda_2 > \cdots > \lambda_r$ then the DSVD is unique.*

## Theorem (D.)

*If $v_1, \dots, v_r$ are t-orthogonal with $t > 2$, then the DSVD is unique.*

# Example: Matrix Multiplication Tensor

$e_1, \ldots, e_n \in \mathbb{C}^n$ are orthogonal
$e_1 \otimes e_1, \ldots, e_n \otimes e_n \in \mathbb{C}^n \otimes \mathbb{C}^n$ are 2-orthogonal

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$e_1 \otimes e_1 \otimes 1, \ldots, e_n \otimes e_n \otimes e_1 \in \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}$ are 2-orthogonal
$e_1 \otimes 1 \otimes e_1, \ldots, e_n \otimes 1 \otimes e_n \in \mathbb{C}^n \otimes \mathbb{C} \otimes \mathbb{C}^n$ are 2-orthogonal
$1 \otimes e_1 \otimes e_1, \ldots, 1 \otimes e_n \otimes e_n \in \mathbb{C} \otimes \mathbb{C}^n \otimes \mathbb{C}^n$ are 2-orthogonal

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Using vertical tensor product, we get

$$\{(e_i \otimes e_j) \otimes (e_j \otimes e_k) \otimes (e_k \otimes e_i) \mid 1 \leq i, j, k \leq n\}$$

are 2-orthogonal.

# Example: Matrix Multiplication Tensor

## Theorem (D.)

*The matrix multiplication tensor*

$$T_n = \sum_{i,j,k=1}^{n} e_{i,j} \otimes e_{j,k} \otimes e_{k,i}$$

*is a DSVD.*

The singular values of $T_n$ are

$$\underbrace{1, 1, \ldots, 1}_{n^3}$$

In particular,

$$\|T_n\|_\star = \sum_{i=1}^{n^3} 1 = n^3.$$

# Example: Group Algebra Multiplication Tensor

$G$ is a group with $n$ elements and $\mathbb{C}G \cong \mathbb{C}^n$ is the group algebra

$$T_G = \sum_{g,h \in G} g \otimes h \otimes h^{-1} g^{-1}.$$

DFT case corresponds to $G = \mathbb{Z}/n\mathbb{Z}$.

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DFT case corresponds to $G = \mathbb{Z}/n\mathbb{Z}$.

**Theorem (D.)**

*$T_G$ has a DSVD and its singular values are*

$$\underbrace{\sqrt{\frac{n}{d_1}}, \dots, \sqrt{\frac{n}{d_1}}}_{d_1^3}, \dots, \underbrace{\sqrt{\frac{n}{d_s}}, \dots, \sqrt{\frac{n}{d_s}}}_{d_s^3}$$

*where $d_1, d_2, \dots, d_s$ are the dimension of the irreducible representations of $G$.*

# $P_n$ no DSVD

Suppose that $P_n$ has a DSVD with singular values $\lambda_1, \ldots, \lambda_r$.

$$\|P_n\|_\star = n^{n/2} = \sum_{i=1}^r \lambda_i$$

$$\|P_n\|_\sigma = \frac{n!}{n^{n/2}} = \lambda_1$$

$$\|P_n\|^2 = n! = \sum_{i=1}^r \lambda_i^2$$

$$\lambda_1 \sum_{i=1}^r \lambda_i = n! = \sum_{i=1}^r \lambda_i^2$$

so $\lambda_1 = \cdots = \lambda_r$, and $r = \frac{r\lambda_1}{\lambda_1} = \frac{\|D_n\|_\star}{\|D_n\|_\sigma} = \frac{n^n}{n!}$.
If $n > 2$ then $r$ is not an integer!

# Noisy Signal Model

Suppose that $V$ is a finite dimensional $\mathbb{R}$-vector space of signals. If $c \in V$ is a noisy signal, then there is a decomposition

$$c = a + b$$

where $a \in V$ is a sparse original signal, and $b \in V$ is additive noise.

Using convex relaxation, sparsity is measured by some $\ell_1$-type norm $\|\cdot\|_X$. The noise is measured by some $\ell_2$-type norm $\|\cdot\|_Y$.

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Using convex relaxation, sparsity is measured by some $\ell_1$-type norm $\|\cdot\|_X$. The noise is measured by some $\ell_2$-type norm $\|\cdot\|_Y$.

For example, $c$ is a tensor in $V = V^{(1)} \otimes \cdots \otimes V^{(d)}$, $\|\cdot\|_X = \|\cdot\|_\star$ is the nuclear norm, and $\|\cdot\|_Y = \|\cdot\|_2 = \|\cdot\|$ is the usual $\ell_2$-norm. (see the CoDe Model)

$V$ finite dimensional $\mathbb{R}$-vector space
$\|\cdot\|_X$, $\|\cdot\|_Y$ norms on $V$
$c \in V$

**Definition**

A pair $(x, y) \in \mathbb{R}^2$ is Pareto-efficient if there exists a decomposition $c = a + b$ with $\|a\| = x$, $\|b\| = y$ and for every decomposition $c = a' + b'$ we have $\|a'\|_X > x$, $\|b'\|_Y > y$ or $(\|a'\|_X, \|b'\|_Y) = (x, y)$. We call $c = a + b$ an $XY$-decomposition. The Pareto-frontier consists of all Pareto-efficient pairs.

# The Pareto Frontier

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$c \in V$

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## Lemma

*The Pareto-frontier is the graph of a strictly decreasing, convex, homeomorphism* $f^c_{YX} : [0, \|c\|_X] \to [0, \|c\|_Y]$.

# The Pareto Sub-Frontier

$\langle \cdot, \cdot \rangle$ positive definite bilinear form
$\|v\|_2 := \|v\| = \sqrt{\langle v, v \rangle}$ $\ell_2$-norm
$\| \cdot \|_X$, $\| \cdot \|_Y$ dual to each others

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**Theorem**

*Equivalent:*

1. $c = a + b$ *is an* $X2$*-decomposition*
2. $c = a + b$ *is an* $2Y$*-decomposition*
3. $\langle a, b \rangle = \|a\|_X \|b\|_Y$.

# The Pareto Sub-Frontier

$\langle \cdot, \cdot \rangle$ positive definite bilinear form
$\|v\|_2 := \|v\| = \sqrt{\langle v, v \rangle}$ $\ell_2$-norm
$\| \cdot \|_X$, $\| \cdot \|_Y$ dual to each others

**Theorem**

*Equivalent:*

1. $c = a + b$ *is an* $X2$*-decomposition*
2. $c = a + b$ *is an* $2Y$*-decomposition*
3. $\langle a, b \rangle = \|a\|_X \|b\|_Y$.

**Definition**

The Pareto sub-frontier consists of all pairs $(x, y)$ such that there exists an $X2$-decomposition $c = a + b$ with $\|a\|_X = x$ and $\|b\|_Y = y$.

# The Pareto Sub-Frontier

**Lemma**

*The Pareto sub-frontier is the graph of a strictly decreasing homeomorphism* $h^c_{YX} : [0, \|c\|_X] \to [0, \|c\|_Y]$.

Clearly, $f^c_{YX} \leq h^c_{YX}$.

**Definition**

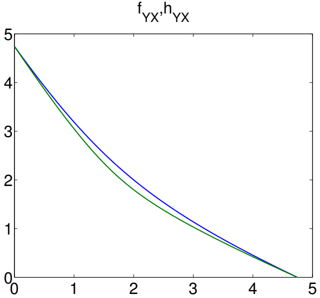
A vector $c \in V$ is called tight if $f^c_{YX} = h^c_{YX}$. If all vectors in $V$ are tight then the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ are called tight.

# (Sub)-Pareto Example

$c = (3\ 3)^t \in V = \mathbb{R}^2$

$$\left\| \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right\|_X = \sqrt{\tfrac{1}{2} z_1^2 + 2 z_2^2}, \quad \left\| \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right\|_Y = \sqrt{2 z_1^2 + \tfrac{1}{2} z_2^2}.$$

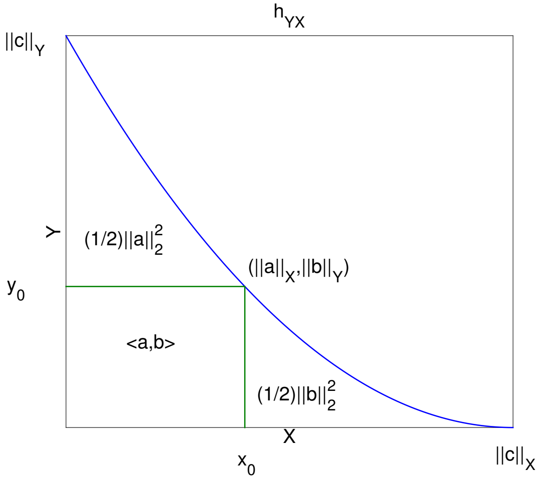
dual norms




So $c$ is not tight.

# The Area Below the Pareto Sub-Frontier

Suppose $c = a + b$ is an $X2$-decomposition with $\|a\|_X = x_0$ and $\|b\|_Y = y_0$. The area under the Pareto sub-frontier is $\frac{1}{2}\|c\|^2 = \frac{1}{2}\|a\|^2 + \langle a, b\rangle + \frac{1}{2}\|b\|^2$.



# The Area Below the Pareto Sub-Frontier

The sub-Pareto frontier applies to:

- denoising a sparse 1D signal: $\|\cdot\|_1$ vs. $\|\cdot\|_\infty$ (tight!)
- denoising of a matrix by soft tresholding: $\|\cdot\|_\star$ vs.$\|\cdot\|_\sigma$ (tight!)
- denoising tensors: $\|\cdot\|_\star$ vs. $\|\cdot\|_\sigma$
- compressive sensing (Basis Pursuit Denoising, LASSO, Dantzig Selector)
- Fatemi-Osher-Rudin image denoising: total variation norm vs. its dual
- $1D$ total variation denoising: Taut String Method

# Example: Singular Value Decomposition

$\mathbb{R}^{n\times m} = \mathbb{R}^n \otimes \mathbb{R}^m$, $r = \min\{n, m\}$
$A \in \mathbb{R}^{n\times m}$ with singular values $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0$

$$\|A\|_X = \|A\|_\star = \lambda_1 + \lambda_2 + \cdots + \lambda_r$$

$$\|A\|_Y = \|A\|_\sigma = \lambda_1$$

$$\|A\| = \sqrt{\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_r^2}$$

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**Lemma**

*The norms $\|\cdot\|_\star$ and $\|\cdot\|_\sigma$ are tight.*

$XY$-decompositions are related to soft tresholding

# Example: Singular Value Decomposition

(note that the Pareto and sub-Pareto frontiers are the same here)

**Lemma**

*If $A$ has singular values $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0$ then the Pareto (sub-)frontier is the piecewise linear function through the points:*

$$(\lambda_1 + \lambda_2 + \cdots + \lambda_k - k\lambda_{k+1}, \lambda_{k+1}), \quad k = 0, 1, 2, \ldots, r$$

*(with the convention $\lambda_{r+1} = 0$).*

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**Corollary**

*The singular values are uniquely determined by the (sub-)Pareto curve, and vice versa.*

# Slope decomposition

## Definition

$c = c_1 + c_2 + \cdots + c_s$ is called a slope decomposition if $\langle c_i, c_j \rangle = \|c_i\|_X \|c_j\|_Y$ for all $i \leq j$ and

$$\frac{\|c_1\|_X}{\|c_1\|_Y} < \frac{\|c_2\|_X}{\|c_2\|_Y} < \cdots < \frac{\|c_r\|_X}{\|c_r\|_Y}.$$

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*$c \in V$ has a slope decomposition if and only if $c$ is tight.*

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**Theorem**

*$c \in V$ has a slope decomposition if and only if $c$ is tight.*

**Theorem**

*The slope decomposition is unique when it exists.*

# Singular values for tight vectors

Suppose $c \in V$ is tight, with slope decomposition $c = c_1 + c_2 + \cdots + c_s$.

**Definition**

Let $\mu_i = \|c_i\|_Y$, and $\lambda_i = \mu_i + \mu_{i+1} + \cdots + \mu_s$ for all $i$. Then $c$ has singular value $\lambda_i$ with multiplicity $\frac{\|c_i\|_X}{\|c_i\|_Y} - \frac{\|c_{i-1}\|_X}{\|c_{i-1}\|_Y}$.

Multiplicities and singular values are positive, but not necessarily integers.

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Multiplicities and singular values are positive, but not necessarily integers.

**Theorem**

*If $c$ has singular Values $\lambda_1, \ldots, \lambda_s$ with multiplicities $m_1, \ldots, m_s$ respectively, then $\|c\|_X = \sum_i m_i \lambda_i$, $\|c\|_Y = \lambda_1$ and $\|T\| = \sqrt{\sum_i m_i \lambda_i^2}$.*

# Singular Values for Tensors

Norms $\|\cdot\|_X = \|\cdot\|_\star$, $\|\cdot\|_Y = \|\cdot\|_\sigma$

**Theorem**

*If $T = \sum_{i=1}^r \lambda_i v_i$ is a diagonal singular value decomposition, then the singular values of $T$ are $\lambda_1, \dots, \lambda_r$.*

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**Theorem**

*If $T = \sum_{i=1}^r \lambda_i v_i$ is a diagonal singular value decomposition, then the singular values of $T$ are $\lambda_1, \ldots, \lambda_r$.*

The permanent tensor $P_n$ is tight, but has no DSVD.
It has singular value $\frac{n!}{n^{n/2}}$ with multiplicity $\frac{n^n}{n!}$.

# Singular Value Region of a Tensor

There is an easy procedure to go from the Pareto subfrontier to the singular values and their multiplicities for tight vectors.

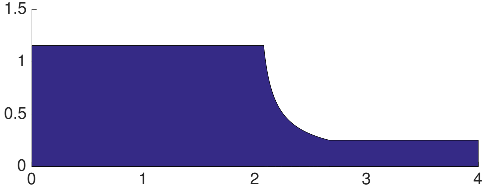
Even if a vector are not tight we can use the same procedure to define a singular spectrum. Some singular valules may have infinitesemal multiplicity.

# Example: Singular Value Region of a Tensor

$\|\cdot\|_X = \|\cdot\|_\star$ and $\|\cdot\|_Y = \|\cdot\|_\sigma$ are dual norms on $\mathbb{R}^{p\times q\times r}$

$$C = e_2 \otimes e_1 \otimes e_1 + e_1 \otimes e_2 \otimes e_1 + e_1 \otimes e_1 \otimes e_2 \in \mathbb{R}^2 \otimes \mathbb{R}^2 \otimes \mathbb{R}^2$$

has continuous singular spectrum




singular value $\frac{2}{\sqrt{3}}$ with multiplicity $\frac{27}{13}$, singular value $\frac{1}{4}$ with multiplicity $\frac{4}{3}$ and singular values between $\frac{1}{4}$ and $\frac{2}{\sqrt{3}}$ with infinitesemal multiplicity
$\|C\|_\sigma = \frac{2}{\sqrt{3}}$ (largest singular value), area is $\|C\|_\star = 3$, and integral of $2y$ over region is $\|C\|_2^2 = 3$

# Thank You!