A complexity theory of constructible sheaves

Symbolic-numeric computing seminar
CUNY

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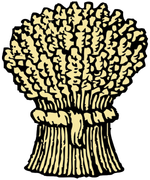

Outline

1. Motivation
2. Qualitative/Background
3. Quantitative/Effective
4. Complexity-theoretic

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Why constructible sheaves ?

- Provides a more natural geometric language, more expressiveness than (first-order) logic.
- It provides a (topological) generalization of quantifier elimination (Tarski-Seidenberg). It is interesting to study quantitative/algorithmic questions in this more general setting.
- Applications in other areas (D-module theory, computational geometry ...).
- Interesting extensions of Blum-Shub-Smale complexity classes leading to $\mathbf{P}$ vs $\mathbf{NP}$ type questions which (paradoxically) might be *easier* to resolve than the classical (B-S-S) ones.
- Quantitative study of sheaf cohomology might be interesting on its own.

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Semi-algebraic sets and maps

- Semi-algebraic sets are subsets of $\mathbb{R}^n$ defined by Boolean formulas whose atoms are polynomial equalities and inequalities (i.e. $P = 0$, $P > 0$ for $P \in \mathbb{R}[X_1, \ldots, X_n]$).
- A semi-algebraic map is a map $X \xrightarrow{f} Y$ between semi-algebraic sets X and Y, is a map whose graph is a semi-algebraic set.

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Closure of semi-algebraic sets under different operations

Easy facts (i.e. follows more-or-less from the definitions) ...

Semi-algebraic sets are closed under:

- Finite unions and intersections, as well as taking complements.
- Products (or more generally fibered products over polynomial maps).
- Taking pull-backs (inverse images) under polynomial maps.

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In the language of arrows instead of quantifiers

- Let $\mathbf{X} \xrightarrow{f} \mathbf{Y}$ be a map (between sets).
- Then there are induced maps:

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\begin{tikzcd}
2^{\mathbf{X}} \arrow[r, shift left=2, "f_{\exists}"] \arrow[r, shift right=2, "f_{\forall}"'] & 2^{\mathbf{Y}} \arrow[l, "f^*"']
\end{tikzcd}
\qquad
\begin{array}{c}
f_{\exists}(A) := f(A) \\
f^*(B) := f^{-1}(B) \\
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\end{array}
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- The pairs $(f_{\exists}, f^*)$ and $(f^*, f_{\forall})$ are not quite pairs of inverses. But ... they do satisfy adjointness relations (namely):

$$f_{\exists} \dashv f^* \dashv f_{\forall}$$

as functors between the poset categories $2^{\mathbf{X}}, 2^{\mathbf{Y}}$ (the objects are subsets and arrows correspond to inclusions).
This is just a *chic* way of saying that for $A \in 2^{\mathbf{X}}, B \in 2^{\mathbf{Y}}$, $f_{\exists}(A) \subset B \Leftrightarrow A \subset f^*(B)$, and $A \subset f^*(B) \Leftrightarrow f_{\forall}(A) \subset B$.

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Tarski-Seidenberg arrow-theoretically ....

- For any semi-algebraic set $\mathbf{X}$, let $\mathcal{S}(\mathbf{X})$ denote the set of semi-algebraic subsets of $\mathbf{X}$.
- Let $\mathbf{X}, \mathbf{Y}$ be semi-algebraic sets, and $\mathbf{X} \xrightarrow{f} \mathbf{Y}$ a polynomial map.
- (Tarski-Seidenberg restated) The restrictions of the maps $f_\exists, f^*, f_\forall$ give functors (maps)

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Triviality of semi-algebraic maps

Yet harder. More than just Tarski-Seidenberg is true...

We say that a semi-algebraic map $\mathbf{X} \xrightarrow{f} \mathbf{Y}$ is semi-algebraically trivial, if there exists $\mathbf{y} \in \mathbf{Y}$, and a semi-algebraic homemorphism $\phi : \mathbf{X} \to \mathbf{X_y} \times \mathbf{Y}$ (denoting $\mathbf{X_y} = f^{-1}(\mathbf{y})$) such that the following diagram is commutative.

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Local triviality of semi-algebraic maps

Theorem (Hardt (1980))

Let $\mathbf{X} \xrightarrow{f} \mathbf{Y}$ *be a semi-algebraic map. Then, there is a finite partition* $\{\mathbf{Y}_i\}_{i \in I}$ *of* $\mathbf{Y}$ *into locally closed semi-algebraic subsets* $\mathbf{Y}_i$*, such that for each* $i \in I$*,* $f|_{f^{-1}(\mathbf{Y}_i)} : f^{-1}(\mathbf{Y}_i) \to \mathbf{Y}_i$ *is semi-algebraically trivial.*

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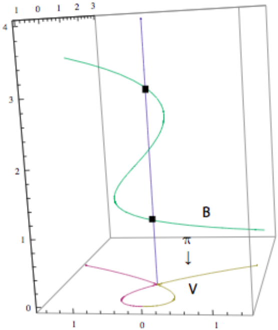

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The formalism of "constructible sheaves" seems to be just the right compromise.

Little detour – Pre-sheaves of A-modules

Let A be a fixed commutative ring. For simplicity we take $A = \mathbb{Q}$.

Definition (Pre-sheaf of A-modules)

A *pre-sheaf* $\mathcal{F}$ of A-modules over a topological space $\mathbf{X}$ associates to each open subset $U \subset \mathbf{X}$ an A-module $\mathcal{F}(\mathbf{U})$, such that that for all pairs of open subsets $\mathbf{U}, \mathbf{V}$ of $\mathbf{X}$, with $\mathbf{V} \subset \mathbf{U}$, there exists a *restriction* homomorphism $r_{\mathbf{U},\mathbf{V}} : \mathcal{F}(\mathbf{U}) \to \mathcal{F}(\mathbf{V})$ satisfying:

1. $r_{\mathbf{U},\mathbf{U}} = \mathrm{Id}_{\mathcal{F}(\mathbf{U})}$;
2. for $\mathbf{U}, \mathbf{V}, \mathbf{W}$ open subsets of $\mathbf{X}$, with $\mathbf{W} \subset \mathbf{V} \subset \mathbf{U}$,

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(For open subsets $\mathbf{U}, \mathbf{V} \subset \mathbf{X}$, $\mathbf{V} \subset \mathbf{U}$, and $s \in \mathcal{F}(\mathbf{U})$, we will sometimes denote the element $r_{\mathbf{U},\mathbf{V}}(s) \in \mathcal{F}(\mathbf{V})$ simply by $s|_{\mathbf{V}}$.)

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Little detour – Pre-sheaves of A-modules

Let A be a fixed commutative ring. For simplicity we take $A = \mathbb{Q}$.

Definition (Pre-sheaf of A-modules)

A *pre-sheaf* $\mathcal{F}$ *of* A*-modules* over a topological space $\mathbf{X}$ associates to each open subset $U \subset \mathbf{X}$ an A-module $\mathcal{F}(\mathbf{U})$, such that that for all pairs of open subsets $\mathbf{U}, \mathbf{V}$ of $\mathbf{X}$, with $\mathbf{V} \subset \mathbf{U}$, there exists a *restriction* homomorphism $r_{\mathbf{U},\mathbf{V}} : \mathcal{F}(\mathbf{U}) \to \mathcal{F}(\mathbf{V})$ satisfying:

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Sheaves with constant coefficients

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1. if $s \in \mathcal{F}(\mathbf{U})$ and $s|_{\mathbf{U}_i} = 0$ for all $i \in I$, then $s = 0$;
2. if for all $i \in I$ there exists $s_i \in \mathcal{F}(\mathbf{U}_i)$ such that

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Stalks of a sheaf

Definition (Stalk of a sheaf at a point)

Let $\mathcal{F}$ be a (pre)-sheaf of A-modules on $\mathbf{X}$ and $\mathbf{x} \in X$. The *stalk* $\mathcal{F}_{\mathbf{x}}$ of $\mathcal{F}$ at $\mathbf{x}$ is defined as the inductive limit

$$\mathcal{F}_{\mathbf{x}} = \varinjlim_{\mathbf{U} \ni \mathbf{x}} \mathcal{F}(\mathbf{U}).$$

Derived category of sheaves on X

- One first considers the category whose objects are *complexes of sheaves* on X, and whose morphisms are *homotopy classes* of morphisms of complexes of sheaves.
- One then localizes with respect to a class of arrows to obtain the derived category $\mathrm{D}(X)$ (resp. $\mathrm{D}^b(X)$).
- This is no longer an abelian category but a *triangulated category*. Exact sequences replaced by distinguished triangles and so on...
- For our purposes it is "ok" to think of an object in $\mathrm{D}(X)$ as a "complex of sheaves".
- If $X = \{\mathrm{pt}\}$, then an object in $\mathrm{D}^b(X)$ is represented by a bounded complex $\mathrm{C}^\bullet$ of A-modules, and $\mathrm{C}^\bullet$ is isomorphic in the derived category to the complex $\mathrm{H}^*(\mathrm{C}^\bullet)$ (with all differentials $= 0$).
- In other words, $\mathrm{C}^\bullet \cong \oplus_{n\in\mathbb{Z}}\mathrm{H}^n(\mathrm{C}^\bullet)[-n]$. But this is *not true* in general (i.e. if X is not a point).

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Operations on sheaves, derived images

Let $\mathcal{F}$ be a sheaf on $\mathbf{X}$, and $\mathcal{G}$ a sheaf on $\mathbf{Y}$, and $f : \mathbf{X} \to \mathbf{Y}$ a continuous map. Then, there exists naturally defined sheaves:

- $f^{-1}(\mathcal{G})$ – a sheaf on $\mathbf{X}$ (pull back). (f^{-1} is an exact functor.)
- The derived direct image denoted $Rf_*(\mathcal{F})$ is an object in $\mathrm{D}(Y)$ (and thus should be thought of as a complex of sheaves on Y).
- We denote for $i \in \mathbb{Z}$, $R^i f_*(\mathcal{F})$ the sheaf $\mathcal{H}^i(Rf_*(\mathcal{F}))$ – but these separately don't determine $Rf_*(\mathcal{F})$.
- In the special case when $\mathcal{F} = A_{\mathbf{X}}$ (the constant sheaf on $\mathbf{X}$), $Rf_*(\mathcal{F})$ is obtained by associating to each open $\mathbf{U} \subset \mathbf{Y}$, a complex of A-modules obtained by taking sections of a flabby resolution of the sheaf $A_{f^{-1}(U)}$.
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High school example – discriminant of a real quadratic

Logical formulation

$$(\exists X) X^2 + 2BX + C = 0$$
$$\Updownarrow$$
$$B^2 - C \geq 0$$

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Geometric formulation

Defining $V \subset \mathbb{R}^3$ (with coordinates X, B, C) defined by $X^2 + 2BX + C = 0$ and $\pi : \mathbb{R}^3 \to \mathbb{R}^2, (x, b, c) \mapsto (b, c)$,

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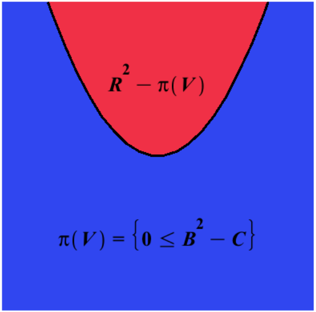

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Sheaf theoretic formulation

Denoting $j : V \hookrightarrow \mathbb{R}^3$, consider the sheaf $j_*(\mathbb{Q}_V) \cong \mathbb{Q}_{\mathbb{R}^3}|_V$, and its (derived) direct image $R\pi_*(j_*(\mathbb{Q}_V))$.

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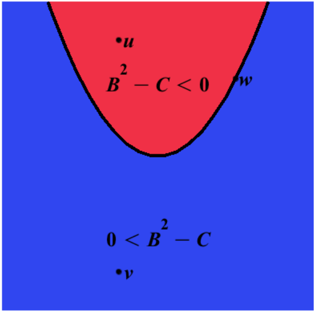

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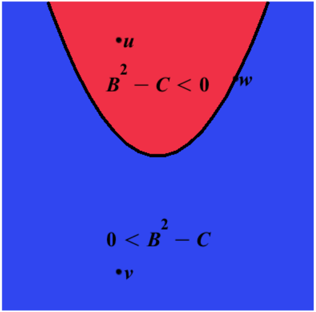

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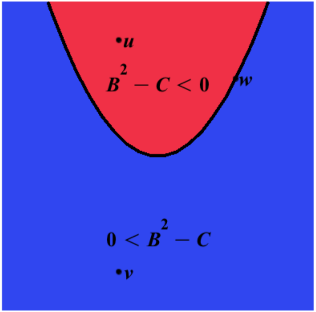

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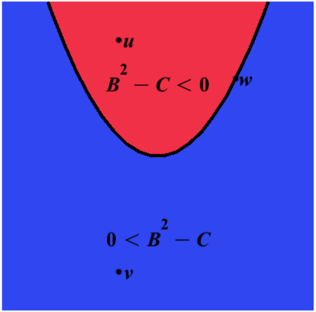

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Constructible sheaves

Definition (Constructible Sheaves)

Let $\mathbf{X}$ be a locally closed semi-algebraic set. Following [Kashiwara-Schapira], an object $\mathcal{F} \in \mathrm{Ob}(\mathbf{D}^b(\mathbf{X}))$ is said to be *constructible* if it satisfies the following two conditions:

(a) There exists a finite partition $\mathbf{X} = \coprod_{i \in I} C_i$ of $\mathbf{X}$ by locally closed semi-algebraic subsets such that for $j \in \mathbb{Z}$ and $i \in I$, the $\mathrm{H}^j(\mathcal{F})|_{C_i}$ are locally constant. We will call such a partition *subordinate* to $\mathcal{F}$.

(b) For each $\mathrm{x} \in \mathbf{X}$, the stalk $\mathcal{F}_{\mathrm{x}}$ has the following properties:

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  (ii) there exists N such that $\mathrm{H}^j(\mathcal{F}_{\mathrm{x}}) = 0$ for all $\mathrm{x} \in \mathbf{X}$ and $|j| > N$.

We will denote the category of constructible sheaves on $\mathbf{X}$ by $\mathbf{D}^b_{\mathrm{sa}}(\mathbf{X})$, and denote by

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(a) There exists a finite partition $\mathbf{X} = \coprod_{i \in I} C_i$ of $\mathbf{X}$ by locally closed semi-algebraic subsets such that for $j \in \mathbb{Z}$ and $i \in I$, the $\mathrm{H}^j(\mathcal{F})|_{C_i}$ are locally constant. We will call such a partition *subordinate* to $\mathcal{F}$.

(b) For each $\mathbf{x} \in \mathbf{X}$, the stalk $\mathcal{F}_{\mathbf{x}}$ has the following properties:
   (i) for each $j \in \mathbb{Z}$, the cohomology groups $\mathrm{H}^j(\mathcal{F}_{\mathbf{x}})$ are finitely generated, and
   (ii) there exists N such that $\mathrm{H}^j(\mathcal{F}_{\mathbf{x}}) = 0$ for all $\mathbf{x} \in \mathbf{X}$ and $|j| > N$.

We will denote the category of constructible sheaves on $\mathbf{X}$ by $\mathbf{D}^b_{\mathrm{sa}}(\mathbf{X})$, and denote by

$$\mathcal{CS}(X) := \mathrm{Ob}(\mathbf{D}^b_{\mathrm{sa}}(\mathbf{X})).$$

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Sheaf-theoretic version of Tarski-Seidenberg

Theorem (Kashiwara (1975), Kashiwara-Schapira (1979))

Let $\mathbf{X} \xrightarrow{f} \mathbf{Y}$ *be a continuous semi-algebraic map. Then for* $\mathcal{F} \in \mathcal{CS}(\mathbf{X})$ *and* $\mathcal{G} \in \mathcal{CS}(\mathbf{Y})$, *then*

$$f^{-1}(\mathcal{G}) \in \mathcal{CS}(\mathbf{X})$$

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More generally, the category of constructible sheaves is closed under the six operations of Grothendieck – namely, $Rf_*, Rf_!, f^{-1}, f^!, \otimes, R\mathcal{H}om$ *– where* f *is a continuous semi-algebraic map.*

End of "Qualitative background". Next, "Quantitative" ....

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Complexity of real quantifier elimination

- Long history, starting with non-elementary-recursive bound of Tarski's original algorithm, doubly exponential algorithm due to Collins (1975) (and also Wuthrich (1976)) using Cylindrical Algebraic Decomposition.
- For each $n \geq 0$, let $\pi_n : \mathbb{R}^n \to \mathbb{R}^{[n/2]}$ denote the projection map forgetting the last $n - [n/2]$ coordinates.
- A new ingredient – *the critical point method* – gives:

Theorem (Grigoriev-Vorobjov (1988), Renegar (1992))

The complexity (both quantitative and algorithmic) of the functors

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Theorem (B. 2014)

The complexity (both quantitative and algorithmic) of the (direct image) functor $R\pi_{n,*} : \mathcal{CS}(\mathbb{R}^n) \to \mathcal{CS}(\mathbb{R}^{[n/2]})$ is bounded singly exponentially.

More precisely:
Let $F \in \mathcal{CS}(\mathbb{R}^n)$ have compact support, and such that there exists a semi-algebraic partition of $\mathbb{R}^n$ subordinate to F defined by the sign conditions on s polynomials of degree at most d, then

(a) there exists a semi-algebraic partition of $\mathbb{R}^{[n/2]}$ subordinate to $R\pi_*(F)$ having complexity $(sd)^{n^{O(1)}}$;

(b) and moreover there exists an algorithm to obtain this partition from the given partition with the same complexity;

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Proof ingredients

- Several ingredients recently developed for studying algorithmic and quantitative questions in semi-algebraic geometry.
- Ideas used to prove singly exponential bounds on the number of homotopy types of the fibers of definable maps (B.-Vorobjov (2007)).
- Singly exponential sized covering by contractibles (B.-Pollack-Roy (2008)).
- Delicate infinitesimal thickening and shrinking arguments.
- Certain arguments using spectral sequences – Leray and Mayer-Vietoris.
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Blum-Shub-Smale complexity classes over $\mathbb{R}$

- Let $\mathcal{S}$ denote the (poset) category of sequences $(S_n \in \mathcal{S}(\mathbb{R}^{m(n)}))_{n>0}$ where each $m(n)$ is a non-negative integer valued function.
- We say that $\mathbf{L} \in \mathcal{S}$ is in $\mathbf{P}_{\mathbb{R}}$, iff there exists a B-S-S machine recognizing $\mathbf{L}$ in polynomial time.
- Recall that we also have sequences of maps:

$$\left(\begin{tikzcd} \mathcal{S}(\mathbb{R}^m) \arrow[r, shift left=2, "\pi_{m,\exists}"] \arrow[r, shift right=2, "\pi_{m,\forall}"'] & \mathcal{S}(\mathbb{R}^{[m/2]}) \arrow[l, "\pi_m^*"'] \end{tikzcd} \right)_{m>0} .$$

- The class $\mathbf{P}_{\mathbb{R}}$ is stable under taking products, unions, intersections, and pull-backs (π_m^*). But what about stability under $\pi_{m,\exists}, \pi_{m,\forall}$?

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$$\left(\mathcal{S}(\mathbb{R}^m) \begin{array}{c} \xrightarrow{\pi_{m,\exists}} \\ \xleftarrow{\pi_m^*} \\ \xrightarrow{\pi_{m,\forall}} \end{array} \mathcal{S}(\mathbb{R}^{[m/2]}) \right)_{m>0}.$$

- The class $\mathbf{P}_{\mathbb{R}}$ is stable under taking products, unions, intersections, and pull-backs (π_m^*). But what about stability under $\pi_{m,\exists}, \pi_{m,\forall}$?

Blum-Shub-Smale complexity classes over $\mathbb{R}$

- Let $\boldsymbol{\mathcal{S}}$ denote the (poset) category of sequences $(S_n \in \mathcal{S}(\mathbb{R}^{m(n)}))_{n>0}$ where each $m(n)$ is a non-negative integer valued function.
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$\mathbf{NP}_{\mathbb{R}}$, $\mathbf{co\text{-}NP}_{\mathbb{R}}$, $\mathbf{PH}_{\mathbb{R}}$ and all that ...

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(Aside) As mentioned before the pairs $(\boldsymbol{\pi}_\exists, \boldsymbol{\pi}^*), (\boldsymbol{\pi}^*, \boldsymbol{\pi}_\forall)$ are not quite pairs of inverse functors, but they form an adjoint triple:

$$\boldsymbol{\pi}_\exists \dashv \boldsymbol{\pi}^* \dashv \boldsymbol{\pi}_\forall.$$

$\mathbf{NP}_{\mathbb{R}}$, $\mathbf{co\text{-}NP}_{\mathbb{R}}$, $\mathbf{PH}_{\mathbb{R}}$ and all that ...

$\pi^*_m, \pi_{m,\exists}, \pi_{m,\forall}$ induce in a natural way the following endo-functors

$$\begin{tikzcd} \mathcal{S} \arrow[r, shift left=2, "\boldsymbol{\pi}_\exists"] \arrow[r, shift right=2, "\boldsymbol{\pi}_\forall"'] & \mathcal{S}. \arrow[l, "\boldsymbol{\pi}^*"'] \end{tikzcd}$$

We have the following obvious inclusions:

$$\mathbf{P}_{\mathbb{R}} \supset \boldsymbol{\pi}^*(\mathbf{P}_{\mathbb{R}}),$$

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$$\mathbf{NP}_{\mathbb{R}} := \boldsymbol{\pi}_\exists(\mathbf{P}_{\mathbb{R}}).$$

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$$\mathbf{PH}_{\mathbb{R}} := \mathbf{P}_{\mathbb{R}} \cup \boldsymbol{\pi}_\exists(\mathbf{P}_{\mathbb{R}}) \cup \boldsymbol{\pi}_\forall(\mathbf{P}_{\mathbb{R}}) \cup \boldsymbol{\pi}_{\exists\forall}(\mathbf{P}_{\mathbb{R}}) \cup \boldsymbol{\pi}_{\forall\exists}(\mathbf{P}_{\mathbb{R}}) \cup \cdots$$

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Complexity classes of constructible sheaves

Definition (Informal definition of the class $\mathcal{P}_{\mathbb{R}}$)

Informally we define the class $\mathcal{P}_{\mathbb{R}}$ as the set of sequences $\left(F_n \in \mathcal{CS}(\mathbb{R}^{m(n)})\right)_{n>0}$ such that

(a) there exists a corresponding sequence of semi-algebraic partitions of $\mathbb{R}^{m(n)}$, subordinate to F_n, in which *point location can be performed efficiently*;

(b) The Poincaré polynomial of the stalks $(F_n)_{\mathbf{x}}, \mathbf{x} \in \mathbb{R}^{m(n)}$ (i.e. the polynomial $P_{(F_n)_{\mathbf{x}}}(T) = \sum_i \dim_i \mathrm{H}^i((F_n)_{\mathbf{x}})T^i$) can be computed efficiently.

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The class $\mathcal{P}_{\mathbb{R}}$ (formally)

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The class $\mathcal{P}_{\mathbb{R}}$ of constructible sheaves consists of sequences $\mathbf{F} = \left(F_n \in \mathcal{CS}(\mathbb{R}^{m(n)})\right)_{n>0}$, where $m(n)$ is a non-negative (polynomially bounded) function satisfying the following conditions. There exists a non-negative (polynomially bounded) function $m_1(n)$ such that:

(a) Each F_n has compact support.

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are computable by B-S-S machines with complexity polynomial in n.

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Example 0

Constant sheaf on compact sequences in $\mathbf{P}_{\mathbb{R}}$

Let $(S_n \in \mathcal{S}(\mathbb{R}^{m(n)}))_{n>0} \in \mathbf{P}^c_{\mathbb{R}}$. Let $j_n : S_n \hookrightarrow \mathbb{R}^n$ be the inclusion map. Then,

$$(j_{n,*}\mathbb{Q}_{S_n})_{n>0} \in \boldsymbol{\mathcal{P}}_{\mathbb{R}}.$$

Stability properties of $\mathcal{P}_{\mathbb{R}}$

Reminiscent of the classical B-S-S complexity class $\mathbf{P}_{\mathbb{R}}$ ...

- The class $\mathcal{P}_{\mathbb{R}}$ is stable under various sheaf operations – direct sums, tensor products, truncation functors.
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- We have the adjunction: $\boldsymbol{\pi}^{-1} \dashv R\boldsymbol{\pi}_*$.
- Similar to the set-theoretic case, the following inclusions can be checked easily.

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Examples of sequences in $\mathbf{\Lambda}_{\mathbb{R}}$

Suppose that $\left(j_n : S_n \hookrightarrow \mathbb{R}^{m(n)}\right)_{n>0}$ belong to $\mathbf{NP}^c_{\mathbb{R}}$ or to $\textbf{co-NP}^c_{\mathbb{R}}$.

Proposition

Then,

$$\left(j_{n,*}\mathbb{Q}_{S_n} \in \mathcal{CS}(\mathbb{R}^{m(n)})\right)_{n>0} \in \mathbf{\Lambda}_{\mathbb{R}}.$$

Conjecture and relation with the classical questions

Conjecture

$$\mathcal{P}_{\mathbb{R}} \neq \mathbf{\Lambda}_{\mathbb{R}}.$$

Theorem (B., 2014)

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In analogy with the set-theoretic case, it is natural to measure the *topological complexity* of a constructible sheaf $F \in \mathcal{CS}(\mathbf{X})$ by

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Complexity theory of constructible functions

Let $\mathbf{X}, \mathbf{Y}$ be compact semi-algebraic sets, and $f : \mathbf{X} \to \mathbf{Y}$ a semi-algebraic continuous map. Then, we have the following commutative diagram:

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\mathcal{CS}(\mathbf{X}) & \underset{f^{-1}}{\overset{Rf_*}{\rightleftarrows}} & \mathcal{CS}(\mathbf{Y}) \\
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where we denote by $\mathrm{CF}(\mathbf{X})$ the set of constructible functions $f : \mathbf{X} \to \mathbb{R}$ on a semi-algebraic set $\mathbf{X}$.

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Open problems and future directions

- Study more precisely the complexity of sheaf operations.
- Get rid of the compactness/properness restrictions or understand better their significance.
- Role of adjointness ? For example, other pairs of adjoint functors such as the pair $(F \overset{L}{\otimes} \cdot \dashv R\mathcal{H}om(\cdot, F))$? More input from abstract category theory ?
- Applications of algorithmic/quantitative sheaf theory in other areas – such as D-modules, algebraic theory of PDE's, computational geometry/topology.
- Study the (simpler) complexity theory of constructible functions instead of sheaves (B-S-S analog of Valiant). This has been developed somewhat including a theory of reduction and complete problems (B. (2014).
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Reference

"A Complexity Theory of Constructible Functions and Sheaves" Saugata Basu, *Foundations of Computational Mathematics*, February 2015, Volume 15, Issue 1, pp 199-279.