Understanding $\delta - \varepsilon$

Simplified definition of a limit: We say that

$$\lim_{x \to a} f(x) = L$$

if we can make the values of $f(x)$ arbitrarily close to L
when we choose some x close enough to a (but not equal to a.)

Goal: We want to force $f(x)$ to be close to L.
We'll do this by choosing x's close to a.

How close to $x = 4$ do we have to be in order to make
$f(x) = \sqrt{x}$ *very close to 2*?

(How about within .1?)

We know that $f(3.61) = 1.9$
and that $f(4.41) = 2.1$,

SO, if we restrict x to be between 3.61 and 4.41,
we **assure** ourselves that $f(x)$ is between 1.9 and 2.1.

Or even if we restrict x to some smaller interval,
$f(x)$ is still between 1.9 and 2.1.

JARGON

δ: A small horizontal (x) value. [`delta`]
ε: A small vertical (y) value. [`epsilon`]

The Official Definition of a Limit:

The limit of $f(x)$ as x approaches a is L when

FOR EVERY EPSILON (ε), we can FIND A DELTA (δ) such that

$$|f(x) - L| < \varepsilon \text{ whenever } 0 < |x - a| < \delta.$$

Restating: No matter how close you want to make $f(x)$ to L, you can always find an x near a to give you that value.

Example: Prove $\lim_{x\to -2}\left(\frac{1}{2}x + 3\right) = 2$ using the $\delta - \varepsilon$ definition of a limit.

Solution: We want to show that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\left|\left(\frac{1}{2}x + 3\right) - 2\right| < \varepsilon \text{ whenever } 0 < |x - (-2)| < \delta.$$

(1) Given $\varepsilon > 0$, what should we guess for δ?
[`Goal: Start out with our` ε `inequality and transform it into our` δ `inequality.`]

$$\begin{aligned}
\left|\frac{1}{2}x + 3 - 2\right| &< \varepsilon \\
\left|\frac{1}{2}x + 1\right| &< \varepsilon \\
\left|\frac{1}{2}x + \frac{1}{2}2\right| &< \varepsilon \\
\left|\frac{1}{2}\right||x + 2| &< \varepsilon \\
|x + 2| &< 2\varepsilon
\end{aligned}$$

So we choose $\delta = 2\varepsilon$.

(2) Showing δ works.
[`We now need to prove the statement we set out to show.`]

Given $\varepsilon > 0$, let $\delta = 2\varepsilon$.
If $0 < |x + 2| < \delta$, [`we ask what does this mean for` $|f(x) - L|$?]
then, $\left|\frac{1}{2}x + 3 - 2\right| = \left|\frac{1}{2}x + 1\right| = \frac{1}{2}|x + 2|$.
But we know $|x + 2| < \delta$, so we can input this information:
$\left|\frac{1}{2}x + 3 - 2\right| < \frac{1}{2}\delta = \frac{1}{2}(2\varepsilon) = \varepsilon$.

So we have showed that whenever $0 < |x - (-2)| < \delta$, we have that $\left|\left(\frac{1}{2}x + 3\right) - 2\right| < \varepsilon$. This is exactly the definition of the limit.