

What else can we do with vectors?

How to multiply two vectors:

$\vec{u} \cdot \vec{v}$ In any dimension: dot product.

What else can we do with vectors?

How to multiply two vectors:

$\vec{u} \cdot \vec{v}$ In any dimension: dot product. Answer is a number.

What else can we do with vectors?

How to multiply two vectors:

$\vec{u} \cdot \vec{v}$ In any dimension: dot product. Answer is a number. Easy.

What else can we do with vectors?

How to multiply two vectors:

$\vec{u} \cdot \vec{v}$ In any dimension: dot product. Answer is a number. Easy.

$\vec{u} \times \vec{v}$ In 3 dimensions: cross product.

What else can we do with vectors?

How to multiply two vectors:

$\vec{u} \cdot \vec{v}$ In any dimension: dot product. Answer is a number. Easy.

$\vec{u} \times \vec{v}$ In 3 dimensions: cross product. Answer is a vector.

What else can we do with vectors?

How to multiply two vectors:

$\vec{u} \cdot \vec{v}$ In any dimension: dot product. Answer is a number. Easy.

$\vec{u} \times \vec{v}$ In 3 dimensions: cross product. Answer is a vector. Memorize.

What else can we do with vectors?

How to multiply two vectors:

$\vec{u} \cdot \vec{v}$ In any dimension: dot product. Answer is a number. Easy.

$\vec{u} \times \vec{v}$ In 3 dimensions: cross product. Answer is a vector. Memorize.

Dot product

Let \vec{a} and \vec{b} be vectors *of the same dimension*.

If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$,

What else can we do with vectors?

How to multiply two vectors:

$\vec{u} \cdot \vec{v}$ In any dimension: dot product. Answer is a number. Easy.

$\vec{u} \times \vec{v}$ In 3 dimensions: cross product. Answer is a vector. Memorize.

Dot product

Let \vec{a} and \vec{b} be vectors *of the same dimension*.

If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, then $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$.

What else can we do with vectors?

How to multiply two vectors:

$\vec{u} \cdot \vec{v}$ In any dimension: dot product. Answer is a number. Easy.

$\vec{u} \times \vec{v}$ In 3 dimensions: cross product. Answer is a vector. Memorize.

Dot product

Let \vec{a} and \vec{b} be vectors *of the same dimension*.

If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, then $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$.

Big deal:

- $\vec{a} \cdot \vec{a} =$

What else can we do with vectors?

How to multiply two vectors:

$\vec{u} \cdot \vec{v}$ In any dimension: dot product. Answer is a number. Easy.

$\vec{u} \times \vec{v}$ In 3 dimensions: cross product. Answer is a vector. Memorize.

Dot product

Let \vec{a} and \vec{b} be vectors *of the same dimension*.

If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, then $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$.

Big deal:

1. $\vec{a} \cdot \vec{a} =$

More Properties:

2. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

What else can we do with vectors?

How to multiply two vectors:

$\vec{u} \cdot \vec{v}$ In any dimension: dot product. Answer is a number. Easy.

$\vec{u} \times \vec{v}$ In 3 dimensions: cross product. Answer is a vector. Memorize.

Dot product

Let \vec{a} and \vec{b} be vectors *of the same dimension*.

If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, then $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$.

Big deal:

1. $\vec{a} \cdot \vec{a} =$

More Properties:

2. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

3. $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

What else can we do with vectors?

How to multiply two vectors:

$\vec{u} \cdot \vec{v}$ In any dimension: dot product. Answer is a number. Easy.

$\vec{u} \times \vec{v}$ In 3 dimensions: cross product. Answer is a vector. Memorize.

Dot product

Let \vec{a} and \vec{b} be vectors *of the same dimension*.

If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, then $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$.

Big deal:

1. $\vec{a} \cdot \vec{a} =$

More Properties:

2. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

3. $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

4. $(c\vec{a}) \cdot \vec{b} = c(\vec{a} \cdot \vec{b})$

What else can we do with vectors?

How to multiply two vectors:

$\vec{u} \cdot \vec{v}$ In any dimension: dot product. Answer is a number. Easy.

$\vec{u} \times \vec{v}$ In 3 dimensions: cross product. Answer is a vector. Memorize.

Dot product

Let \vec{a} and \vec{b} be vectors *of the same dimension*.

If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, then $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$.

Big deal:

1. $\vec{a} \cdot \vec{a} =$

More Properties:

2. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

3. $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

4. $(c\vec{a}) \cdot \vec{b} = c(\vec{a} \cdot \vec{b})$

5. $\vec{0} \cdot \vec{a} = \underline{\quad}$

Dot products and angles

Key idea: Use the dot product to find the angle between vectors.

Dot products and angles

Key idea: Use the dot product to find the angle between vectors.

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = |\vec{\mathbf{a}}| |\vec{\mathbf{b}}| \cos \theta \quad \text{OR} \quad \cos \theta = \frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}}{|\vec{\mathbf{a}}| |\vec{\mathbf{b}}|}.$$

Dot products and angles

Key idea: Use the dot product to find the angle between vectors.

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = |\vec{\mathbf{a}}| |\vec{\mathbf{b}}| \cos \theta \quad \text{OR} \quad \cos \theta = \frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}}{|\vec{\mathbf{a}}| |\vec{\mathbf{b}}|}.$$

Why?

Dot products and angles

Key idea: Use the dot product to find the angle between vectors.

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = |\vec{\mathbf{a}}| |\vec{\mathbf{b}}| \cos \theta \quad \text{OR} \quad \cos \theta = \frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}}{|\vec{\mathbf{a}}| |\vec{\mathbf{b}}|}.$$

Why? Law of cosines!!

Dot products and angles

Key idea: Use the dot product to find the angle between vectors.

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = |\vec{\mathbf{a}}| |\vec{\mathbf{b}}| \cos \theta \quad \text{OR} \quad \cos \theta = \frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}}{|\vec{\mathbf{a}}| |\vec{\mathbf{b}}|}.$$

Why? Law of cosines!! $|\vec{\mathbf{a}} - \vec{\mathbf{b}}|^2 = |\vec{\mathbf{a}}|^2 + |\vec{\mathbf{b}}|^2 - 2 |\vec{\mathbf{a}}| |\vec{\mathbf{b}}| \cos \theta$

Dot products and angles

Key idea: Use the dot product to find the angle between vectors.

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = |\vec{\mathbf{a}}| |\vec{\mathbf{b}}| \cos \theta \quad \text{OR} \quad \cos \theta = \frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}}{|\vec{\mathbf{a}}| |\vec{\mathbf{b}}|}.$$

Why? Law of cosines!! $|\vec{\mathbf{a}} - \vec{\mathbf{b}}|^2 = |\vec{\mathbf{a}}|^2 + |\vec{\mathbf{b}}|^2 - 2 |\vec{\mathbf{a}}| |\vec{\mathbf{b}}| \cos \theta$

Example. What is the angle between $\vec{\mathbf{a}} = \langle 2, 2, -1 \rangle$ and $\vec{\mathbf{b}} = \langle 5, -3, 2 \rangle$?

Answer:

Dot products and angles

Key idea: Use the dot product to find the angle between vectors.

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = |\vec{\mathbf{a}}| |\vec{\mathbf{b}}| \cos \theta \quad \text{OR} \quad \cos \theta = \frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}}{|\vec{\mathbf{a}}| |\vec{\mathbf{b}}|}.$$

Why? Law of cosines!! $|\vec{\mathbf{a}} - \vec{\mathbf{b}}|^2 = |\vec{\mathbf{a}}|^2 + |\vec{\mathbf{b}}|^2 - 2 |\vec{\mathbf{a}}| |\vec{\mathbf{b}}| \cos \theta$

Example. What is the angle between $\vec{\mathbf{a}} = \langle 2, 2, -1 \rangle$ and $\vec{\mathbf{b}} = \langle 5, -3, 2 \rangle$?

Answer:

$$\cos^{-1} \left(\frac{2}{3\sqrt{38}} \right) \approx 1.46 \text{ rad} \approx 84^\circ.$$

Dot products and angles

Key idea: Use the dot product to find the angle between vectors.

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = |\vec{\mathbf{a}}| |\vec{\mathbf{b}}| \cos \theta \quad \text{OR} \quad \cos \theta = \frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}}{|\vec{\mathbf{a}}| |\vec{\mathbf{b}}|}.$$

Why? Law of cosines!! $|\vec{\mathbf{a}} - \vec{\mathbf{b}}|^2 = |\vec{\mathbf{a}}|^2 + |\vec{\mathbf{b}}|^2 - 2 |\vec{\mathbf{a}}| |\vec{\mathbf{b}}| \cos \theta$

Example. What is the angle between $\vec{\mathbf{a}} = \langle 2, 2, -1 \rangle$ and $\vec{\mathbf{b}} = \langle 5, -3, 2 \rangle$?

Answer:

$$\cos^{-1} \left(\frac{2}{3\sqrt{38}} \right) \approx 1.46 \text{ rad} \approx 84^\circ.$$

Question: What happens when two vectors are orthogonal?

Key idea: Two vectors are orthogonal if and only if _____.

Projecting

Dot products let you project one vector onto another.

Answers: “How far does vector $\vec{\mathbf{b}}$ go in vector $\vec{\mathbf{a}}$ ’s direction?”

Projecting

Dot products let you project one vector onto another.

Answers: “How far does vector $\vec{\mathbf{b}}$ go in vector $\vec{\mathbf{a}}$ ’s direction?”

First: Calculate the length of the projection.

Draw the triangle.

We see $\frac{|\text{proj}_{\vec{\mathbf{a}}}\vec{\mathbf{b}}|}{|\vec{\mathbf{b}}|} = \cos \theta = \underline{\hspace{2cm}},$

Projecting

Dot products let you project one vector onto another.

Answers: “How far does vector $\vec{\mathbf{b}}$ go in vector $\vec{\mathbf{a}}$ ’s direction?”

First: Calculate the length of the projection.

Draw the triangle.

We see $\frac{|\text{proj}_{\vec{\mathbf{a}}}\vec{\mathbf{b}}|}{|\vec{\mathbf{b}}|} = \cos \theta = \text{_____},$

So its length is $|\text{proj}_{\vec{\mathbf{a}}}\vec{\mathbf{b}}| = \frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}}{|\vec{\mathbf{a}}|}.$

Projecting

Dot products let you project one vector onto another.

Answers: “How far does vector $\vec{\mathbf{b}}$ go in vector $\vec{\mathbf{a}}$ ’s direction?”

First: Calculate the length of the projection.

Draw the triangle.

We see $\frac{|\text{proj}_{\vec{\mathbf{a}}}\vec{\mathbf{b}}|}{|\vec{\mathbf{b}}|} = \cos \theta = \underline{\hspace{2cm}},$

So its length is $|\text{proj}_{\vec{\mathbf{a}}}\vec{\mathbf{b}}| = \frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}}{|\vec{\mathbf{a}}|}.$

Next: What is the direction of the projection?

The unit vector in $\vec{\mathbf{a}}$ ’s direction is $\underline{\hspace{2cm}}.$

Projecting

Dot products let you project one vector onto another.

Answers: “How far does vector $\vec{\mathbf{b}}$ go in vector $\vec{\mathbf{a}}$ ’s direction?”

First: Calculate the length of the projection.

Draw the triangle.

We see $\frac{|\text{proj}_{\vec{\mathbf{a}}}\vec{\mathbf{b}}|}{|\vec{\mathbf{b}}|} = \cos \theta = \underline{\hspace{2cm}}$,

So its length is $|\text{proj}_{\vec{\mathbf{a}}}\vec{\mathbf{b}}| = \frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}}{|\vec{\mathbf{a}}|}$.

Next: What is the direction of the projection?

The unit vector in $\vec{\mathbf{a}}$ ’s direction is $\underline{\hspace{2cm}}$.

Therefore

$$\text{proj}_{\vec{\mathbf{a}}}\vec{\mathbf{b}} = \frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}}{|\vec{\mathbf{a}}|} \cdot \frac{\vec{\mathbf{a}}}{|\vec{\mathbf{a}}|} = \frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}}{|\vec{\mathbf{a}}|^2} \vec{\mathbf{a}}$$

Cross Products

3D Only!!!!

Cross Products

3D Only!!!!

Given vectors $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, the cross product:

$$\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

Cross Products

3D Only!!!!

Given vectors $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, the cross product:

$$\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

is *orthogonal* to both \vec{a} and \vec{b}

Cross Products

3D Only!!!!

Given vectors $\vec{\mathbf{a}} = \langle a_1, a_2, a_3 \rangle$ and $\vec{\mathbf{b}} = \langle b_1, b_2, b_3 \rangle$, the cross product:

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

is *orthogonal* to both $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ and has length

$$|\vec{\mathbf{a}} \times \vec{\mathbf{b}}| = |\vec{\mathbf{a}}| |\vec{\mathbf{b}}| \sin \theta.$$

Cross Products

3D Only!!!!

Given vectors $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, the cross product:

$$\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

is *orthogonal* to both \vec{a} and \vec{b} and has length

$$|\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin \theta.$$

This is equal to the area of the parallelogram determined by \vec{a} and \vec{b} .

Cross Products

3D Only!!!!

Given vectors $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, the cross product:

$$\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

is *orthogonal* to both \vec{a} and \vec{b} and has length

$$|\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin \theta.$$

This is equal to the area of the parallelogram determined by \vec{a} and \vec{b} .

Use the right hand rule to determine the direction of $\vec{a} \times \vec{b}$.

- ▶ Use your *right hand* to swing from \vec{a} to \vec{b} .
Your thumb points in the direction of $\vec{a} \times \vec{b}$.

Cross Products

3D Only!!!!

Given vectors $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, the cross product:

$$\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

is *orthogonal* to both \vec{a} and \vec{b} and has length

$$|\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin \theta.$$

This is equal to the area of the parallelogram determined by \vec{a} and \vec{b} .

Use the right hand rule to determine the direction of $\vec{a} \times \vec{b}$.

- ▶ Use your *right hand* to swing from \vec{a} to \vec{b} .
Your thumb points in the direction of $\vec{a} \times \vec{b}$.

(What do you get?)

Remembering $\langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$

Use the determinant of a 3×3 matrix.

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Example. Find $\langle 2, 3, 2 \rangle \times \langle 1, 0, 6 \rangle$, and show that it is \perp to each.

Properties of \times

- ▶ $\vec{a} \times \vec{a} = \vec{0}$
- ▶ $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
- ▶ $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
- ▶ $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$
- ▶ $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

Properties of \times

- ▶ $\vec{a} \times \vec{a} = \vec{0}$
- ▶ $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
- ▶ $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
- ▶ $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$
- ▶ $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

Proofs by component manipulation

$$\vec{a} \times (\vec{b} + \vec{c}) =$$

Properties of \times

- ▶ $\vec{a} \times \vec{a} = \vec{0}$
- ▶ $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
- ▶ $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
- ▶ $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$
- ▶ $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

Proofs by component manipulation

$$\begin{aligned}\vec{a} \times (\vec{b} + \vec{c}) &= \\ &= \langle a_1, a_2, a_3 \rangle \times (\langle b_1, b_2, b_3 \rangle + \langle c_1, c_2, c_3 \rangle)\end{aligned}$$

Properties of \times

- ▶ $\vec{a} \times \vec{a} = \vec{0}$
- ▶ $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
- ▶ $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
- ▶ $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$
- ▶ $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

Proofs by component manipulation

$$\begin{aligned}\vec{a} \times (\vec{b} + \vec{c}) &= \\ &= \langle a_1, a_2, a_3 \rangle \times (\langle b_1, b_2, b_3 \rangle + \langle c_1, c_2, c_3 \rangle) \\ &= \langle a_1, a_2, a_3 \rangle \times \langle b_1 + c_1, b_2 + c_2, b_3 + c_3 \rangle\end{aligned}$$

Properties of \times

$$\blacktriangleright \vec{a} \times \vec{a} = \vec{0}$$

$$\blacktriangleright \vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

$$\blacktriangleright \vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

$$\blacktriangleright \vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$$

$$\blacktriangleright \vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$

Proofs by component manipulation

$$\vec{a} \times (\vec{b} + \vec{c}) =$$

$$= \langle a_1, a_2, a_3 \rangle \times (\langle b_1, b_2, b_3 \rangle + \langle c_1, c_2, c_3 \rangle)$$

$$= \langle a_1, a_2, a_3 \rangle \times \langle b_1 + c_1, b_2 + c_2, b_3 + c_3 \rangle$$

$$\langle a_2(b_3 + c_3) - a_3(b_2 + c_2), a_3(b_1 + c_1) - a_1(b_3 + c_3), a_1(b_2 + c_2) - a_2(b_1 + c_1) \rangle$$

Properties of \times

$$\blacktriangleright \vec{a} \times \vec{a} = \vec{0}$$

$$\blacktriangleright \vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

$$\blacktriangleright \vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

$$\blacktriangleright \vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$$

$$\blacktriangleright \vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$

Proofs by component manipulation

$$\vec{a} \times (\vec{b} + \vec{c}) =$$

$$= \langle a_1, a_2, a_3 \rangle \times (\langle b_1, b_2, b_3 \rangle + \langle c_1, c_2, c_3 \rangle)$$

$$= \langle a_1, a_2, a_3 \rangle \times \langle b_1 + c_1, b_2 + c_2, b_3 + c_3 \rangle$$

$$\langle a_2(b_3 + c_3) - a_3(b_2 + c_2), a_3(b_1 + c_1) - a_1(b_3 + c_3), a_1(b_2 + c_2) - a_2(b_1 + c_1) \rangle +$$

$$= \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle +$$

Properties of \times

$$\blacktriangleright \vec{a} \times \vec{a} = \vec{0}$$

$$\blacktriangleright \vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

$$\blacktriangleright \vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

$$\blacktriangleright \vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$$

$$\blacktriangleright \vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$

Proofs by component manipulation

$$\vec{a} \times (\vec{b} + \vec{c}) =$$

$$= \langle a_1, a_2, a_3 \rangle \times (\langle b_1, b_2, b_3 \rangle + \langle c_1, c_2, c_3 \rangle)$$

$$= \langle a_1, a_2, a_3 \rangle \times \langle b_1 + c_1, b_2 + c_2, b_3 + c_3 \rangle$$

$$\langle a_2(b_3 + c_3) - a_3(b_2 + c_2), a_3(b_1 + c_1) - a_1(b_3 + c_3), a_1(b_2 + c_2) - a_2(b_1 + c_1) \rangle$$

$$= \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle +$$

$$\langle a_2 c_3 - a_3 c_2, a_3 c_1 - a_1 c_3, a_1 c_2 - a_2 c_1 \rangle$$

Properties of \times

$$\blacktriangleright \vec{a} \times \vec{a} = \vec{0}$$

$$\blacktriangleright \vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

$$\blacktriangleright \vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

$$\blacktriangleright \vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$$

$$\blacktriangleright \vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$

Proofs by component manipulation

$$\vec{a} \times (\vec{b} + \vec{c}) =$$

$$= \langle a_1, a_2, a_3 \rangle \times (\langle b_1, b_2, b_3 \rangle + \langle c_1, c_2, c_3 \rangle)$$

$$= \langle a_1, a_2, a_3 \rangle \times \langle b_1 + c_1, b_2 + c_2, b_3 + c_3 \rangle$$

$$= \langle a_2(b_3 + c_3) - a_3(b_2 + c_2), a_3(b_1 + c_1) - a_1(b_3 + c_3), a_1(b_2 + c_2) - a_2(b_1 + c_1) \rangle$$

$$= \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle +$$

$$\langle a_2 c_3 - a_3 c_2, a_3 c_1 - a_1 c_3, a_1 c_2 - a_2 c_1 \rangle$$

$$= \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

Properties of \times

$$\blacktriangleright \vec{a} \times \vec{a} = \vec{0}$$

$$\blacktriangleright \vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

$$\blacktriangleright \vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

$$\blacktriangleright \vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$$

$$\blacktriangleright \vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$

Proofs by component manipulation

$$\vec{a} \times (\vec{b} + \vec{c}) =$$

$$= \langle a_1, a_2, a_3 \rangle \times (\langle b_1, b_2, b_3 \rangle + \langle c_1, c_2, c_3 \rangle)$$

$$= \langle a_1, a_2, a_3 \rangle \times \langle b_1 + c_1, b_2 + c_2, b_3 + c_3 \rangle$$

$$= \langle a_2(b_3 + c_3) - a_3(b_2 + c_2), a_3(b_1 + c_1) - a_1(b_3 + c_3), a_1(b_2 + c_2) - a_2(b_1 + c_1) \rangle$$

$$= \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle +$$

$$\langle a_2 c_3 - a_3 c_2, a_3 c_1 - a_1 c_3, a_1 c_2 - a_2 c_1 \rangle$$

$$= \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

The quantity $|\vec{a} \cdot (\vec{b} \times \vec{c})|$ is called the **scalar triple product**, and calculates the volume of the *parallelepiped* determined by the vectors \vec{a} , \vec{b} , and \vec{c} .

Physics

Application: Work

If a force applied in a direction (vector $\vec{\mathbf{F}}$) causes a displacement in a direction (vector $\vec{\mathbf{D}}$), then the work exerted is $W = \vec{\mathbf{F}} \cdot \vec{\mathbf{D}}$.

Physics

Application: Work

If a force applied in a direction (vector \vec{F}) causes a displacement in a direction (vector \vec{D}), then the work exerted is $W = \vec{F} \cdot \vec{D}$.

Application: Torque

If a force applied in a direction (vector \vec{F}) is applied to a lever, where the radius vector \vec{r} is from the pivot to the place where the force is applied, then a turning force called **torque** $\vec{\tau}$ is generated. A formula is calculated by: $\vec{\tau} = \vec{r} \times \vec{F}$