

Lines, Planes, and Automobiles!

Lines in 2D Coordinates

Two common formats:

$$y = mx + b \text{ (slope-intercept) or } (y - y_0) = m(x - x_0) \text{ (pt-slope)}$$

Given a point and a direction, you know the equation of the line.

$$\langle x, y \rangle = \langle x_0, y_0 \rangle + t \langle a, b \rangle$$

Two Lines

In three dimensions, two lines can

- ▶ be parallel
- ▶ intersect
- ▶ be skew

Lines in 3D Coordinates

$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle$$

Each t gives a point (x, y, z) on L .

Reading componentwise, same as:

$$\begin{cases} x(t) = x_0 + at \\ y(t) = y_0 + bt \\ z(t) = z_0 + ct \end{cases}$$

Key idea: Read off direction vector \vec{v} from coeffs of t .

1D Examples

Example. Find the equation of the line that passes through $A = (2, 4, -3)$ and $B = (3, -1, 1)$.

Answer: To find the equation of a line, we need

- ▶ One Point.
- ▶ One Direction.

Example. Where does this line pass through the xy -plane?

Answer: In other words, _____.

Never the twain shall meet

Example. Show that the following lines are skew.

$$\text{Romeo : } \langle 1 + t, -2 + 3t, 4 - t \rangle$$

$$\text{Juliet : } \langle 2s, 3 + s, -3 + 4s \rangle$$

Answer: We will show:

- ▶ They are not **parallel**. (They would have the same _____.)

- ▶ They do not **intersect**. (There would be a point _____.)

Equations of planes

Question:

Does **a plane**
have a direction?

There is one vector _____ to the plane, the _____ \vec{n} .

Note: \vec{n} defines *infinitely many* planes. We also need a point.

A **plane** is defined by a normal vector \vec{n} and a point $\vec{r}_0 = (x_0, y_0, z_0)$.

For any point \vec{r} on the plane, $\vec{r} - \vec{r}_0$ is perpendicular to \vec{n} .

So the equation of a plane is

Alternate forms

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0.$$

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$ax + by + cz = d$$

Key idea: Read off
normal vector \vec{n}
from coeffs of x, y, z .

Plane Examples

Example. What is the angle between the planes

$$x + y + z = 1 \quad \text{and} \quad x - 2y + 3z = 1?$$

Answer: When we need to find an angle, use _____.

$$\theta = \cos^{-1} \left(\frac{2}{\sqrt{42}} \right) \approx 72^\circ$$

Example. What is the equation of the intersection line?

Answer: For the equation of a line, we need _____.

Plane Examples

Example. Find the distance from $(1, 0, -1)$ to $2x + 3y - 5z + 10 = 0$.

Answer: The normal vector to the plane is _____, so the shortest distance from $P_0 = (1, 0, -1)$ to the plane is along the line $(1, 0, -1) + t(2, 3, -5)$. Where does this hit the plane?

Use the equation of the line and the plane:

$$2x + 3y - 5z + 10 = 0 \rightsquigarrow 2(1 + 2t) + 3(0 + 3t) - 5(-1 - 5t) + 10 = 0$$

Simplifying, the point P_1 where the line hits the plane is when $t = \frac{-17}{38}$.

$$\vec{P}_1 - \vec{P}_0 = \frac{-17}{38} \langle 2, 3, -5 \rangle, \text{ so}$$

$$|\vec{P}_1 - \vec{P}_0| = \frac{17}{38} \sqrt{2^2 + 3^2 + (-5)^2} = \frac{17}{\sqrt{38}}.$$