Functions

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Vector functions

$\vec{\mathbf{r}} : \mathbb{R} \to \mathbb{R}^3$ (or $\mathbb{R}^2$ or $\mathbb{R}^n$)

$\vec{\mathbf{r}} : t \mapsto \langle f(t), g(t), h(t)\rangle$

$\vec{\mathbf{r}}$ takes in a real number t

outputs a vector $\langle f(t), g(t), h(t)\rangle$

Limities and Helices

The **limit** of a vector function $\vec{\mathbf{r}}$ is defined by taking the limits of its component functions

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Plug in some values of t

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Derivatives and Derivative-derivative definitions

Define $\vec{\mathbf{r}}'(t) = \lim_{h \to 0} \frac{\vec{\mathbf{r}}(t+h) - \vec{\mathbf{r}}(t)}{h} = \left\langle f'(t), g'(t), h'(t) \right\rangle$.

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Remember: Indefinite integrals have a (vector) constant of integration.

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Example. $\int \langle 2\cos t, \sin t, 2t\rangle\,dt = \langle 2\sin t, -\cos t, t^2\rangle + \vec{\mathbf{C}}$.

Derivatives

Example. Find the equation of the tangent line to the helix $\vec{\mathbf{r}}(t) = \langle 2\cos t, \sin t, t \rangle$ at the point $P = (0, 1, \frac{\pi}{2})$.

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Game plan:

1. Find t^* for which the curve goes through the point P.
2. Find the tangent vector $\vec{\mathbf{r}}'(t)$, plug in $t = t^*$.
3. Write the equation of the line.

Derivatives rule

- $\frac{d}{dt}\big(\vec{\mathbf{r}}(t) + \vec{\mathbf{s}}(t)\big) = \vec{\mathbf{r}}'(t) + \vec{\mathbf{s}}'(t)$
- $\frac{d}{dt}\big(c\,\vec{\mathbf{r}}(t)\big) = c\vec{\mathbf{r}}'(t)$
- $\frac{d}{dt}\big(f(t)\,\vec{\mathbf{r}}(t)\big) = f'(t)\,\vec{\mathbf{r}}(t) + f(t)\,\vec{\mathbf{r}}'(t)$
- $\frac{d}{dt}\big(\vec{\mathbf{r}}(t) \cdot \vec{\mathbf{s}}(t)\big) = \vec{\mathbf{r}}'(t) \cdot \vec{\mathbf{s}}(t) + \vec{\mathbf{r}}(t) \cdot \vec{\mathbf{s}}'(t)$
- $\frac{d}{dt}\big(\vec{\mathbf{r}}(t) \times \vec{\mathbf{s}}(t)\big) = \vec{\mathbf{r}}'(t) \times \vec{\mathbf{s}}(t) + \vec{\mathbf{r}}(t) \times \vec{\mathbf{s}}'(t)$
- $\frac{d}{dt}\big(\vec{\mathbf{r}}(f(t))\big) = f'(t)\,\vec{\mathbf{r}}'(f(t))$

Motion in space

If $\vec{\mathbf{r}}(t)$ is the vector position of a particle, then

- $\vec{\mathbf{r}}'(t) = \vec{\mathbf{v}}(t)$ is the vector velocity of the particle.
- $|\vec{\mathbf{r}}'(t)| = |\vec{\mathbf{v}}(t)| =$ speed of the particle.
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Example. Suppose that a mass of 40 kg starts with init. pos'n $\langle 1, 0, 0\rangle$, initial velocity $\langle 1, -1, 1\rangle$ and has acceleration $\vec{\mathbf{a}}(t) = \langle 4t, 6t, 1\rangle$.

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(a) Find the position and velocity of the particle as a function of t.

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Example. Show that if a particle moves with constant speed, then the velocity and acceleration vectors are orthogonal.