

## More partials

This works with more variables too.

$$\frac{\partial}{\partial z} (e^{xy} \ln z) = \text{_____} \quad \text{and} \quad \frac{\partial}{\partial x} (e^{xy} \ln z) = \text{_____}$$

We can also take higher derivatives.

$$\frac{\partial}{\partial x} \frac{\partial}{\partial x} f(x, y) \quad \text{or} \quad \frac{\partial^2}{\partial x^2} f(x, y) \quad \text{or} \quad f_{xx}(x, y)$$

We might even decide to **mix** our partial derivatives.

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y} \frac{\partial}{\partial x} f(x, y).$$

## A big deal: Partial Differential Equations

- ▶ Laplace's Equation:  $\frac{\partial^2}{\partial x^2} u(x, y) + \frac{\partial^2}{\partial y^2} u(x, y) = 0$  is a PDE.
  - ▶ Solutions (fcns  $u$  that satisfy) give formulas related to distribution of heat on a surface, how fluids & electricity flow.
- ▶ Wave Equation:  $\frac{\partial^2}{\partial t^2} u(x, t) = a \frac{\partial^2}{\partial x^2} u(x, t)$  is a PDE.
  - ▶ Solutions describe the position of waves as a function of time.

# Clairaut's Theorem

**Example.** Calculate all second-order partial derivatives of

$$f(x, y) = x^3 + x^2y^3 - 2y^2.$$

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 $f_x =$

$f_y =$

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 $f_{xx} =$

$f_{yx} =$

$f_{xy} =$

$f_{yy} =$ 


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Notice: \_\_\_\_\_

**Clairaut's Theorem** (mid 1700's)

Suppose  $f(x, y)$  is defined on a disk  $D$  containing  $(a, b)$ .

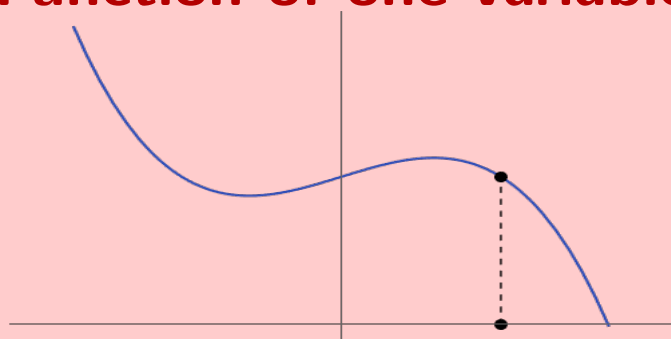
**If**  $f_{xy}$  and  $f_{yx}$  are continuous on  $D$ , **then**  $f_{xy}(a, b) = f_{yx}(a, b)$ .

**Consequence:** Order partial derivatives however you want.

$$f_{xyzz} = f_{zxyz} = f_{zyzx} = \dots$$

# Interpretation of partial derivatives

## Function of one variable



$$\frac{d}{dx} f(x) \quad \text{at} \quad x = a$$

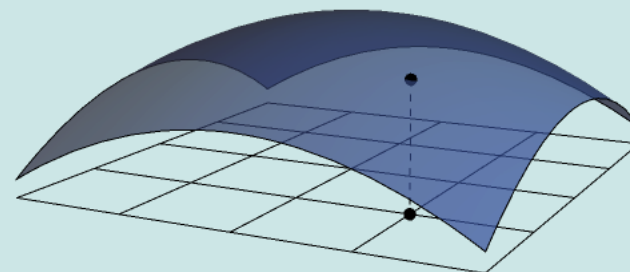
slope of tangent line to the curve

$$y = f(x)$$

at  $x = a$ .

“What is the rate of change of  $f(x)$  as  $x$  changes?”

## Function of several variables



$$\frac{\partial}{\partial x} f(x, y) \quad \text{at} \quad (x, y) = (a, b)$$

slope of tangent line to the curve  
on the surface  $z = f(x, y)$  where  
sliced by the vertical plane  $y = b$   
at  $x = a$ .

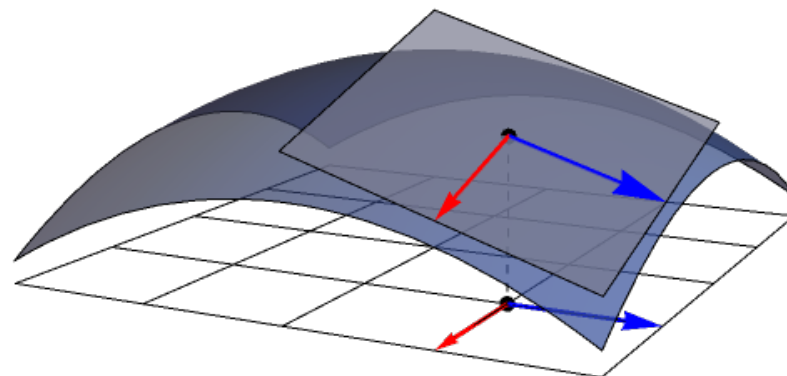
“If  $y$  is fixed, what is the rate of change of  $f(x, y)$  as  $x$  changes?”

# Tangency

$\frac{\partial f}{\partial x}$  and  $\frac{\partial f}{\partial y}$  at  $(x_0, y_0)$  are slopes of tangent lines along the surface.

They lie in the **tangent plane**.

“When we zoom into a smooth surface, the surface looks like a plane.”



*For any curve* on the surface through  $(x_0, y_0, f(x_0, y_0))$ , the tangent line to the curve would be in this plane too.

**Key Idea:** When the tangent plane to the surface is a good approximation for the the graph when  $(x, y)$  is near  $(a, b)$ , we say that  $f$  is **differentiable** at  $(a, b)$ .

**Theorem.** If the partial derivatives  $f_x$  and  $f_y$  **exist** near  $(a, b)$  and **are continuous** at  $(a, b)$ , then  $f$  is differentiable at  $(a, b)$ .

# Planey

The equation of this tangent plane is easy. (point-slope)

$$(z - z_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

**Example.** What is the eqn of the tangent plane to  $z = xe^{xy}$  at  $(1, 0)$ ?

## Game Plan:

1. Find the partial derivatives, evaluate at  $(1, 0)$ . (Find slopes)
2. Determine the point on the surface. (Find point)
3. Write down equation of plane. (Point-slope formula)

# Linear Approximation

**Key idea:** Use tangent plane as linear approximation to the function.

$T(x, y)$  gives a “good enough” value for  $f(x, y)$  near  $(x_0, y_0)$ .

**Example.** Use a linear approximation of  $f(x, y) = xe^{xy}$  to approximate  $f(1.1, -0.1)$ .

**Answer:**  $(1.1, -0.1)$  is a point near \_\_\_\_\_.

The tangent plane there is  $T(x, y) = x + y$ .

So:  $f(1.1, -0.1) \approx T(1.1, -0.1) =$  \_\_\_\_\_.

Note:  $f(1.1, -0.1) = 1.1e^{-0.11} \approx .985$ .

**In more dimensions:** Suppose  $f(\vec{v}) = f(w, x, y, z)$ .

A linear approximation near  $\vec{v}_0 = (w_0, x_0, y_0, z_0)$  would be

$$f(\vec{v}) - f(\vec{v}_0) \approx f_w(\vec{v}_0)(w - w_0) + f_x(\vec{v}_0)(x - x_0) + f_y(\vec{v}_0)(y - y_0) + f_z(\vec{v}_0)(z - z_0)$$

# Differentials

Differentials are the other way to understand linear approximations.  
How much does  $z$  change as  $x$  and  $y$  change?

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$dz$  is a *approximation* for how much  $z$  actually changes.

**Example.** If  $z = x^2 + 3xy - y^2$ , find  $dz$ .

$$dz =$$

**Conclusion:** If  $x$  changes from  $2 \rightarrow 2.05$ ,  $dx =$  \_\_\_\_\_

If  $y$  changes from  $3 \rightarrow 2.96$ ,  $dy =$  \_\_\_\_\_.

We would expect  $z$  to change by \_\_\_\_\_.

The true change is .6449.