

Definition of the directional derivative

Partial derivatives allow us to see how fast a function changes.

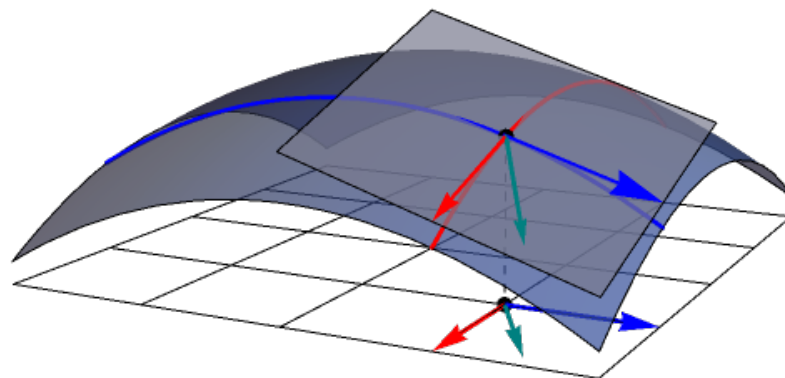
$D_x f = f_x(x, y)$ is the rate of change of f in the x -direction. Toward $\vec{i} = (1, 0)$

$D_y f = f_y(x, y)$ is the rate of change of f in the y -direction. Toward $\vec{j} = (0, 1)$

Question: How fast is $f(x, y)$ changing in **some other direction**?

What does that even mean?

Question: What is the rate of change of f toward unit vector $\vec{u} = (a, b) = (\cos \theta, \sin \theta)$?



Definition: The directional derivative of f in the direction of \vec{u} is

$$D_{\vec{u}} f(x, y) = f_x(x, y) a + f_y(x, y) b.$$

Directional derivative example

Example. Find $D_{\vec{u}}f$ if $f(x, y) = x^3 - 3xy + 4y^2$ and \vec{u} is the unit vector in the xy -plane at angle $\theta = \pi/6$.

Answer: First, find the vector $\vec{u} =$
Next, find the partial derivatives:

$$\frac{\partial f}{\partial x} = \qquad \qquad \qquad \frac{\partial f}{\partial y} =$$

We conclude that $D_{\vec{u}}f(x, y) =$

Example. Calculate $D_{\vec{u}}f(1, 2)$ and interpret this answer.

$$\begin{aligned} D_{\vec{u}}f(1, 2) &= (3 \cdot 1 - 3 \cdot 2) \frac{\sqrt{3}}{2} + (-3 \cdot 1 + 8 \cdot 2) \frac{1}{2} \\ &= \frac{13 - 2\sqrt{3}}{2} \approx 3.9 \end{aligned}$$

Interpretation: One unit step in the \vec{u} direction increases $f(x, y)$ by approximately 3.9 units.

Motivating the gradient

Notice that $D_{\vec{u}}f = f_x a + f_y b$.

Rewrite as: $D_{\vec{u}}f = \langle f_x, f_y \rangle \cdot \langle a, b \rangle$.

Definition: The vector $\langle f_x, f_y \rangle = f_x \vec{i} + f_y \vec{j}$ is called the **gradient** of f . We write ∇f or **grad** f .

So an alternate way to write $D_{\vec{u}}f(x, y)$ is $\nabla f(x, y) \cdot \vec{u}$.

The gradient is also defined for functions of more than two variables. For example, for a function of three variables, $f(x, y, z)$,

$$\nabla f = \langle f_x, f_y, f_z \rangle = f_x \vec{i} + f_y \vec{j} + f_z \vec{k}$$

$$\text{and } D_{\vec{u}}f = \nabla f \cdot \vec{u}$$

Applying ∇f

Example. Let $f(x, y, z) = x \sin(yz)$. Find the directional derivative of f at $(1, 3, 0)$ in the direction $\vec{v} = \vec{i} + 2\vec{j} - \vec{k}$.

Step back. What do we want to calculate?

Game Plan:

- ▶ Find a unit vector in the direction of \vec{v} .
- ▶ Find ∇f , plug in $(1, 3, 0)$.
- ▶ Take the dot product.

Therefore $D_{\vec{u}}f(1, 3, 0) =$

Interpretation?

An important interpretation of the gradient

Question: Given a function $f(x, y)$ and a point (x_0, y_0) ,
(or a function $f(x, y, z)$ and a point (x_0, y_0, z_0)),
in which direction is the function increasing the *fastest*?
And how fast is the function increasing in that direction?

Answer: At a rate of $|\nabla f(x_0, y_0)|$, in the direction of $\nabla f(x_0, y_0)$!!

But why?!?
$$D_{\vec{u}}f = \nabla f \cdot \vec{u} = |\nabla f| |\vec{u}| \cos(\theta)$$
$$= |\nabla f| \cos(\theta)$$

Question: For what angle θ is this maximized? And what is the max?

Answer:

Consequence: ∇f represents the direction of fastest increase of f .

Visualization of the gradient

∇f represents the direction of fastest increase of f .

We can understand this graphically through the contour map.

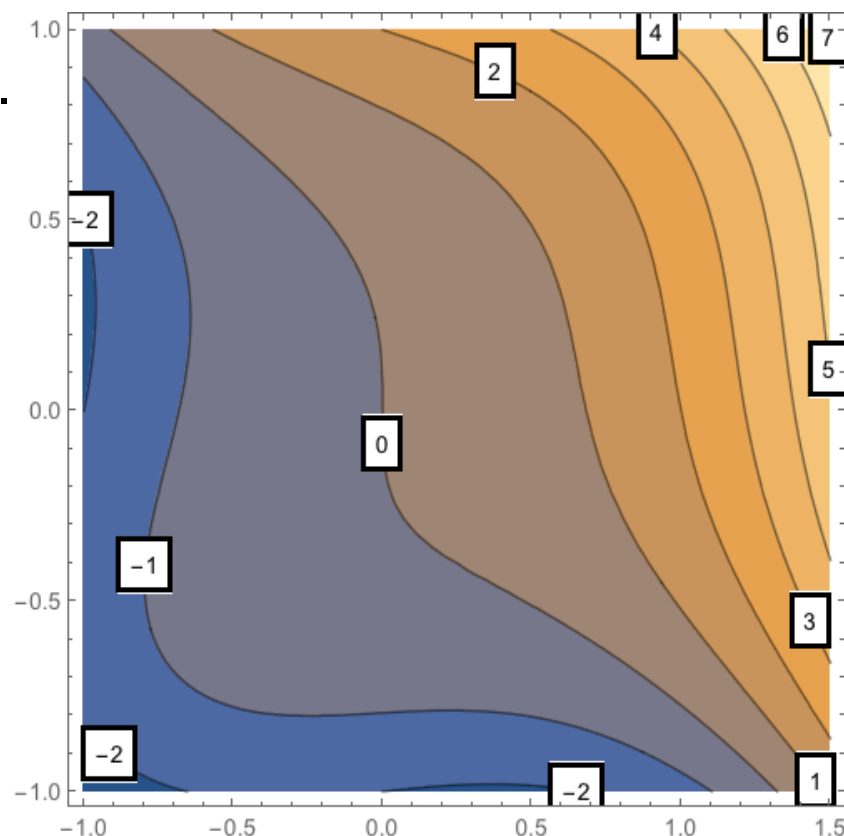
- ▶ At (x_0, y_0) , the vector $\nabla f(x_0, y_0)$ is perpendicular to the level curves of f .

Why?

- ▶ Along a level curve, f is constant.
- ▶ The fastest change should be perpendicular to the level curve.

♡ Connecting along this path gives ♡
♡ the **path of steepest ascent**. ♡

Chloe says “hi”.



Tangent planes to level surfaces

Functions of two variables

A *level curve* $f(x, y) = c$

$\nabla f \longleftrightarrow$ fastest increase

So: ∇f is \perp (to tangent line)
to level curve at (x_0, y_0)

Functions of three variables

A *level surface* $F(x, y, z) = c$

$\nabla F \longleftrightarrow$ fastest increase

so ∇F is \perp (to tangent plane)
to level surface at (x_0, y_0, z_0)

$\nabla F(x_0, y_0, z_0)$ is the **normal vector** to the level surface at (x_0, y_0, z_0) .

This means: The equation of THE **tangent plane** to
THE **level surface** passing through the point (x_0, y_0, z_0) is

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

Also: For any curve $\vec{r}(t) = (x(t), y(t), z(t))$ on the level surface,

$$F(x(t), y(t), z(t)) = k \xrightarrow{\text{chain}} \frac{\partial F}{\partial x} \frac{dx}{dt} + \frac{\partial F}{\partial y} \frac{dy}{dt} + \frac{\partial F}{\partial z} \frac{dz}{dt} = 0,$$

which means $\nabla F \perp \vec{r}'(t) = 0$.