Local Extrema

Functions of one variable

$f(x)$ has a **local maximum** at $x=a$ if for all points x near a, $f(x)\leq f(a)$.
$f(x)$ has a **local minimum** at $x=a$ if for all points x near a, $f(x)\geq f(a)$.

(What about global/absolute?)

If $f(x)$ has a local max or local min at $x=a$, **then:**

$f'(a)=0$ or $f'(a)$ does not exist.

A point a where this is true is called a **critical point**.

However, **If** $f'(a)=0$ or $f'(a)$ DNE **then** this **does not imply** that $x=a$ is a local max or min.

Functions of multiple variables

$f(x,y)$ has a **local maximum** at (a,b) if for all points nearby, $f(x,y)\leq f(a,b)$.
$f(x,y)$ has a **local minimum** at (a,b) if for all points nearby, $f(x,y)\geq f(a,b)$.

(What about global/absolute?)

If $f(x,y)$ has a local max or local min at $(x,y)=(a,b)$, **then:**

$f_x(a,b)=0$ and $f_y(a,b)=0$ (or DNE)

A point (a,b) where this is true is called a **critical point**.

However, **If** (f_x and f_y) $=0$ or DNE **then** this **does not imply** that $(x,y)=(a,b)$ is a local max or min.

Determining local extrema

Important vocabulary:

- A maximum or minimum: means ________________.
- A maximum value or minimum value: means ________.

We can try to determine if a critical point is a local extremum using:

The second derivative test.

If the second partial derivatives of $f(x, y)$ are continuous around (a, b)
And if $f_x(a, b) = 0$ and $f_y(a, b) = 0$, **then** define $D(a, b)$:

$$D(a, b) = f_{xx}(a, b) \cdot f_{yy}(a, b) - \left(f_{xy}(a, b)\right)^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$

1. If $D > 0$ and $f_{xx} > 0$, then (a, b) is a local minimum.
2. If $D > 0$ and $f_{xx} < 0$, then (a, b) is a local maximum.
3. If $D < 0$, then (a, b) is a **saddle point** of f.
4. If $D = 0$, the test is inconclusive.

Extreme Examples

Example. Find the local extrema and saddle points of

$$f(x, y) = x^4 + y^4 - 4xy + 1.$$

- Critical points:

- For each: Find $D(a, b)$, classify.

Global Extrema

Functions of one variable	Functions of multiple variables
Extreme Value Theorem: If f is continuous on a **closed interval**, then f attains an absolute max and absolute min **somewhere** on this interval.	**Extreme Value Theorem:** If f is continuous on a __________ **set** in $\mathbb{R}^2$, then f attains an absolute max and absolute min **somewhere** on this set.

Example. Find the global extrema of $f(x, y) = x^2 - 2xy + 2y$ on the rectangle $0 \leq x \leq 3$ and $0 \leq y \leq 2$.

- Check for crit. pts. on interior. $\left.\begin{array}{l} 0 = f_x = 2x - 2y \\ 0 = f_y = -2x + 2 \end{array}\right\} \to (1, 1)$
- Find extreme values on boundary.

  $f(x, 0) =$ __________ $\rightsquigarrow$ min:__________ max:__________

  $f(3, y) =$ __________ $\rightsquigarrow$ min:__________ max:__________

  $f(x, 2) =$ __________ $\rightsquigarrow$ min:__________ max:__________

  $f(0, y) =$ __________ $\rightsquigarrow$ min:__________ max:__________
- Determine largest & smallest.

Optimization is just finding maxima and minima

Example. A rectangular box with no lid is made from $12\,\mathrm{m}^2$ of cardboard. What is the maximum volume of the box?

Solution. Let length, width, and height be x, y, and z, respectively. Then the question asks us to maximize $V =$ __________, subject to ________________.

Solving for z gives $z = \frac{12-xy}{2x+2y}$. Inserting, $V = xy\left(\frac{12-xy}{2x+2y}\right)$.

To find an optimum value, solve for $\frac{\partial V}{\partial x} = 0$ and $\frac{\partial V}{\partial x} = 0$.

$\frac{\partial V}{\partial x} = 0 \rightsquigarrow$

$\frac{\partial V}{\partial y} = 0 \rightsquigarrow$

Solving these simultaneous equations, $12 - 2xy = x^2 = y^2 \Rightarrow x = \pm y$. Because this is real world, ______, so we solve $12 - 3x^2 = 0$: ______.

This problem must have an absolute maximum, which must occur at a critical point. (Why?) Therefore $(x, y, z) = (2, 2, 1)$ is the absolute maximum, and the maximum volume is $xyz = 4$.

Optimization subject to constraints

The method of Lagrange multipliers is an alternative way to find maxima and minima of a function $f(x, y, z)$ subject to a given constraint $g(x, y, z) = k$.

Motivating Example. Suppose you are trying to find the maximum and minimum value of $f(x, y) = y - x$ when we only consider points on the curve $g(x, y) = x^2 + 4y^2 = 36$.

What should we do?