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Important vocabulary:

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4. If $D = 0$, the test is inconclusive.

Extreme Examples

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- Critical points:

- For each: Find $D(a, b)$, classify.

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Example. Find the global extrema of $f(x, y) = x^2 - 2xy + 2y$ on the rectangle $0 \le x \le 3$ and $0 \le y \le 2$.

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- Find extreme values on boundary.

  $f(x, 0) =$ __________ $\rightsquigarrow$ min:__________ max:__________
  $f(3, y) =$ __________ $\rightsquigarrow$ min:__________ max:__________
  $f(x, 2) =$ __________ $\rightsquigarrow$ min:__________ max:__________
  $f(0, y) =$ __________ $\rightsquigarrow$ min:__________ max:__________
- Determine largest & smallest.

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This problem must have an absolute maximum, which must occur at a critical point. (Why?) Therefore $(x, y, z) = (2, 2, 1)$ is the absolute maximum, and the maximum volume is $xyz = 4$.

Optimization subject to constraints

The method of Lagrange multipliers is an alternative way to find maxima and minima of a function $f(x, y, z)$ subject to a given constraint $g(x, y, z) = k$.

Motivating Example. Suppose you are trying to find the maximum and minimum value of $f(x, y) = y - x$ when we only consider points on the curve $g(x, y) = x^2 + 4y^2 = 36$.

What should we do?