Riemann Sum Comparison

Riemann sum to approximate area

Subdivide $[a, b]$ into n intervals I.

Interval width: $\Delta x_i = x_i - x_{i-1}$.

Choose sample point x_i^* in each I

The Riemann sum $\sum_{i=1}^{n} f(x_i^*)\Delta x_i$

approximates area under curve.

Integral is limit of Riemann sums.

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

(If exists)

Riemann Sum to approximate volume

(Today: Rectangles $[a, b] \times [c, d]$.)

Subdivide $[a, b]$ into m intervals.
Subdivide $[c, d]$ into n intervals.

Each subrect. $[x_{i-1}, x_i] \times [y_{j-1}, y_j]$ has area $\Delta A_{ij} = \Delta x_i \Delta y_j$.

The Riemann sum $\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_i^*)\Delta A_{ij}$

approximates volume under surface.

The **double integral** is limit of **R.S.**

$$\iint_R f(x, y)\,dA = \lim_{\substack{\max \Delta x_i \to 0 \\ \Delta y_i \to 0}} \text{R.S.}$$

(If exists)

Notes and an example

- If f is continuous, the limit always exists!
- The volume under the surface is approximated by adding up the volumes of a bunch of rectangular prisms.

Example. Estimate the volume that lies above the square $R = [0, 2] \times [0, 2]$ and below the surface $z = 16 - x^2 - 2y^2$ by calculating a Riemann sum with four terms.

Answer: The region R can be divided into four squares, each with area $\Delta A =$ _____.

Choose a sample point in each region.
(Ex: Upper right corner.)

$$\text{Volume} \approx f(1,1)\Delta A_{11} + f(1,2)\Delta A_{12} + f(2,1)\Delta A_{21} + f(2,2)\Delta A_{22}$$
$$=$$

Calculating double integrals

We never calculate a double integral by taking limits of Riemann sums.

Instead, we use the method of iterated integrals:

Calculate the volume by adding slices together.
For a fixed x, we can find the area of an (infinitesimal) slice of $f(x, y)$ by integrating over y. Then we can integrate these areas over all slices to give the volume.

Area of slice: $A(x) = \int_{y=c}^{y=d} f(x, y)\,dy$ *[Think of x as constant.]*

Volume: $V = \int_{x=a}^{x=b} A(x)\,dx = \int_{x=a}^{x=b} \left[\int_{y=c}^{y=d} f(x, y)\,dy \right] dx.$

We write $\int_a^b \int_c^d f(x, y)\,dy\,dx.$

The order of the dy and dx tells you which to integrate first.
Work from the inside out.

Fubini's Theorem

If f is continuous on the rectangle $R = [a, b] \times [c, d]$ then

$$\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy$$

Also true in general if

- f is bounded on R
- f discontinuous only on a finite number of smooth curves
- the iterated integrals exist

Intuition: In our picture, we could have sliced the volume by fixing y instead of x.

Take away message: When f is nice, we can choose the order of integration to make our life easier.

Double integrals

Example. Find $\iint_R y\sin(xy)\,dA$ where $R = [1,2]\times[0,\pi]$.

Is it easier to integrate with respect to _____.

Properties of double integrals

When $f(x,y)$ is a product of (a fcn of x) and (a fcn of y) over a rectangle $[a,b]\times[c,d]$, then the double integral decomposes nicely:

$$\iint_R g(x)h(y)\,dA = \left[\int_a^b g(x)\,dx\right]\cdot\left[\int_c^d h(y)\,dy\right]$$

For general regions R: (not necessarily rectangles)

- $\iint_R (f+g)\,dA = \iint_R f\,dA + \iint_R g\,dA$
- $\iint_R cf\,dA = c\iint_R f\,dA$
- If $f(x,y)\geq g(x,y)$ for all $(x,y)\in R$, then $\iint_R f\,dA \geq \iint g\,dA$.