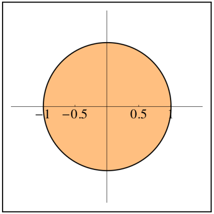
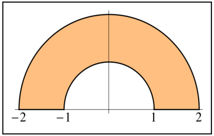
Some regions *demand* polar coordinates

Some regions are best described in polar coordinates:



Parts of circles



Parts of annuli

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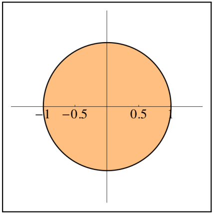

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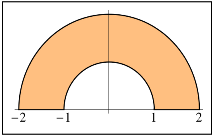

Parts of annuli

These are called **polar rectangles** because they have the form

$$\left\{ \begin{array}{c} a \le r \le b \\ \alpha \le \theta \le \beta \end{array} \right\} \text{ for constants } \left\{ \begin{array}{c} a,\ b \\ \alpha,\ \beta \end{array} \right\}$$

Polar coordinates

Goal: Convert a double integral involving x's and y's
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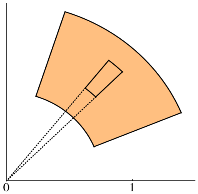

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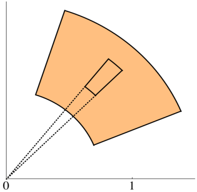

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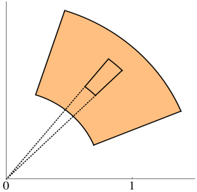

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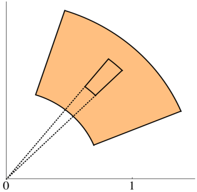

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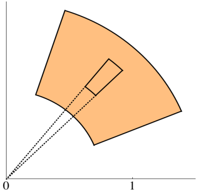

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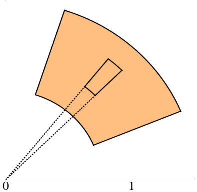

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Consequence: To calculate $\iint_D f(x,y)\,dA$, when D is best described in polar coordinates, calculate

$$\iint_D f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

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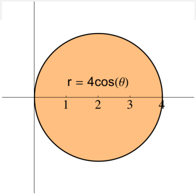

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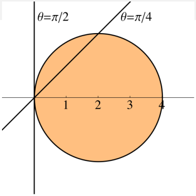

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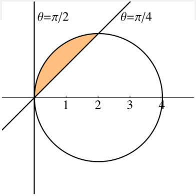

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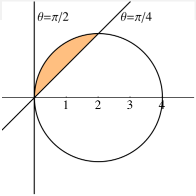

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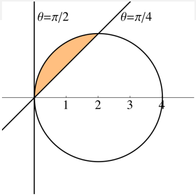

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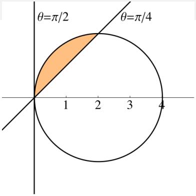

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$$\cdots = \int_{\theta=0}^{\theta=\pi} K\frac{a^3}{3}\, d\theta = \left[K\frac{a^3}{3}\theta\right]_{\theta=0}^{\theta=\pi} = \frac{\pi K a^3}{3}.$$

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Example. Calculate $\int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (x^3 + xy^2)\, dy\, dx$

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