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- Regions are not boxes. (Today)
- Regions are best defined in polar-like coordinates. (Next time)

Setting up complicated triple integrals

- We only consider triple integrals over regions that can be defined as being between two surfaces.

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You may need to try multiple projections to find the easiest integral to integrate. Then use all the tools in your toolbox to integrate it.

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Now calculate $\iint_{\substack{D=\text{circle}\\ x^2+z^2=4}} \left(\int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\, dy \right) dA$

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There is no y, so the innermost integral is easy:

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This integral is easier to do using ________________

Loose ends

Density in three dimensions

- Given a mass density function $\rho(x, y, z)$ (mass per unit volume)

$$\text{mass} = \iiint_E \rho(x, y, z)\, dV.$$

Average value in three dimensions

- The average value of a function $f(x, y, z)$ over a region E is

$$f_{ave} = \frac{1}{V(E)} \iiint_E f(x, y, z)\, dV.$$