#### Area under a parametric curve

Given y = f(x), the area under the curve from x = a to x = b is Area  $= \int_{x=a}^{x=b} \leftarrow \text{right endpoint} = \int_{t=\alpha}^{t=\beta} \leftarrow \text{right endpoint} = \int_{t=\alpha}^{t=\beta} \frac{g(t)f'(t)dt}{g(t)f'(t)dt}$ 

Example. Find the area under one arch of the cycloid  $\begin{cases} x = r(\theta - \sin \theta) \\ y = r(1 - \cos \theta) \end{cases}$ (Here, *r* is a constant and  $\theta$  is the parameter.)

Plot it to see the shape. One arch has range  $\_\_\_ \leq \theta \leq \_\_\_$ .

$$A = \int y \, dx = \int r(1 - \cos \theta) r(1 - \cos \theta) \, d\theta$$
$$= r^2 \int (1 - 2\cos \theta + \cos^2 \theta) \, d\theta$$

### Area inside a polar curve

For cartesian functions y = f(x), calculate area as  $A = \int dA = \int y \, dx$ .

What about the area "*inside a curve*" best described as a polar function  $r = f(\theta)$ ?

• We still use 
$$A = \int dA$$
.

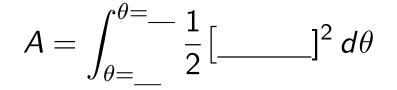
But the formula for dA is different.

$$A = \int_{\theta=a}^{\theta=b} \frac{1}{2} \left[ f(\theta) \right]^2 d\theta = \int_a^b \frac{1}{2} r^2 d\theta$$

**Polar land** How much area is swept out by a little slice?

Example. What is the area inside one loop of the four-leaved rose  $r = \cos 2\theta$ ?

- ▶ What are the bounds on  $\theta$ ?
- For which θ does the curve pass through the origin?



## Inside yet Outside

**Example.** Calculate the area inside the curve  $r = 3 \sin \theta$  and outside the curve  $r = \sin \theta + 1$ .

First: Draw a picture!

- What are these curves?
- ► Where do they intersect?

Now calculate:  $\int_{\alpha_{-}}^{\theta=} \left[ \left( \frac{1}{2} (3\sin\theta)^2 \right) - \left( \frac{1}{2} (\sin\theta + 1)^2 \right) \right] d\theta$  $=\frac{1}{2}\int_{\theta=}^{\theta=}$   $\left[9\sin^2\theta - (\sin^2\theta + 2\sin\theta + 1)\right]d\theta$  $= \frac{1}{2} \int_{\theta}^{\theta} \frac{1}{2} \left[ 8 \sin^2 \theta - 2 \sin \theta - 1 \right] d\theta$  $=\frac{1}{2}\int_{\theta}^{\theta} \left[4-4\cos 2\theta-2\sin \theta-1\right]d\theta$  $=\frac{1}{2}\int_{\theta=}^{\theta=} [3-4\cos 2\theta - 2\sin \theta] d\theta$  $=\left[\frac{3\theta}{2}-\sin 2\theta+\cos \theta\right]_{\pi/6}^{5\pi/6}$  $=\frac{3}{2}\left(\frac{5\pi}{6}-\frac{\pi}{6}\right)-\left(\sin\frac{10\pi}{6}-\sin\frac{2\pi}{6}\right)+\left(\cos\frac{5\pi}{6}-\cos\frac{\pi}{6}\right)=\cdots$  $=\pi + \sqrt{3} - \sqrt{3} = \pi$ 

## Arc length of a parametric curve

To find the arc length of a parametric curve, think  $L = \int dL$ . How much arc length  $\Delta L$  does the curve traverse in one time unit  $\Delta t$ ?

$$\Delta L_{i} = \sqrt{(\Delta x_{i})^{2} + (\Delta y_{i})^{2}} = \sqrt{(f'(t_{i}^{*})\Delta t_{i})^{2} + (g'(t_{i}^{**})\Delta t_{i})^{2}}$$
(there is some  $t_{i}^{*}$  and some  $t_{i}^{**}$  in the time interval...)
$$L = \lim_{n \to \infty} \sum_{i=1}^{n} \Delta L_{i} = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{(f'(t_{i}^{*}))^{2} + (g'(t_{i}^{**}))^{2}} \Delta t_{i}$$
(Similar to a Riemann Sum)
$$L = \int_{t=\alpha}^{t=\beta} \sqrt{[f'(t)]^{2} + [g'(t)]^{2}} dt$$
Example. Find the arc length for the parametric curve 
$$\begin{cases} x = \sin 2t \\ y = \cos 2t \end{cases}$$
for  $0 \le t \le 2\pi$ . What do we expect? What is this curve?

$$\int_{t=0}^{t=2\pi} \sqrt{\left[f'(t)
ight]^2 + \left[g'(t)
ight]^2} \, dt =$$

# Arc length of a polar curve

To calculate arc length, view a polar curve as a parametric curve.

• Convert 
$$r = f(\theta)$$
 to  $\begin{cases} x = f(\theta) \cos \theta \\ y = f(\theta) \sin \theta \end{cases}$   
• Then  $L = \int_{\theta=\alpha}^{\theta=\beta} \sqrt{\left[\frac{d}{d\theta}(f(\theta)\cos\theta)\right]^2 + \left[\frac{d}{d\theta}(f(\theta)\sin\theta)\right]^2} d\theta$ 

... take derivatives & do the algebra ...

• 
$$L = \int_{\theta=\alpha}^{\theta=\beta} \sqrt{\left[f(\theta)\right]^2 + \left[\frac{d}{d\theta}(f(\theta))\right]^2} d\theta$$

We can also understand this as  $dL = \sqrt{(r \, d\theta)^2 + (dr)^2}$