What else can we do with vectors?

How to multiply two vectors:

$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}$ In any dimension: dot product. Answer is a number. Easy.

$\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ In 3 dimensions: cross product. Answer is a vector. Memorize.

Dot product

Let $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ be vectors *of the same dimension*.

If $\vec{\mathbf{a}} = \langle a_1, a_2, a_3 \rangle$ and $\vec{\mathbf{b}} = \langle b_1, b_2, b_3 \rangle$, then $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = a_1 b_1 + a_2 b_2 + a_3 b_3$.

Big deal:

1. $\vec{\mathbf{a}} \cdot \vec{\mathbf{a}} =$

More Properties:

2. $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \vec{\mathbf{b}} \cdot \vec{\mathbf{a}}$
3. $\vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} + \vec{\mathbf{c}}) = \vec{\mathbf{a}} \cdot \vec{\mathbf{b}} + \vec{\mathbf{a}} \cdot \vec{\mathbf{c}}$
4. $(c\vec{\mathbf{a}}) \cdot \vec{\mathbf{b}} = c(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})$
5. $\vec{\mathbf{0}} \cdot \vec{\mathbf{a}} =$ __

Dot products and angles

Key idea: Use the dot product to find the angle between vectors.

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = |\vec{\mathbf{a}}|\,|\vec{\mathbf{b}}|\,\cos\theta \qquad \text{OR} \qquad \cos\theta = \frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}}{|\vec{\mathbf{a}}|\,|\vec{\mathbf{b}}|}.$$

Why? Law of cosines!! $\left|\vec{\mathbf{a}} - \vec{\mathbf{b}}\right|^2 = |\vec{\mathbf{a}}|^2 + \left|\vec{\mathbf{b}}\right|^2 - 2\,|\vec{\mathbf{a}}|\,\left|\vec{\mathbf{b}}\right|\cos\theta$

Example. What is the angle between $\vec{\mathbf{a}} = \langle 2, 2, -1\rangle$ and $\vec{\mathbf{b}} = \langle 5, -3, 2\rangle$?
Answer:

$$\cos^{-1}\left(\frac{2}{3\sqrt{38}}\right) \approx 1.46 \text{ rad} \approx 84^\circ.$$

Question: What happens when two vectors are orthogonal?

Key idea: Two vectors are orthogonal if and only if ____________.

Projecting

Dot products let you project one vector onto another.

Answers: "How far does vector $\vec{\mathbf{b}}$ go in vector $\vec{\mathbf{a}}$'s direction?"

First: Calculate the length of the projection.

Draw the triangle.

We see $\dfrac{\left|\mathrm{proj}_{\vec{\mathbf{a}}}\vec{\mathbf{b}}\right|}{|\vec{\mathbf{b}}|} = \cos\theta =$ ______________,

So its length is $\left|\mathrm{proj}_{\vec{\mathbf{a}}}\vec{\mathbf{b}}\right| = \frac{\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}}{|\vec{\mathbf{a}}|}$.

Next: What is the direction of the projection?

The unit vector in $\vec{\mathbf{a}}$'s direction is ________.

Therefore

$$\mathrm{proj}_{\vec{\mathbf{a}}}\vec{\mathbf{b}} = \frac{\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}}{|\vec{\mathbf{a}}|}\cdot\frac{\vec{\mathbf{a}}}{|\vec{\mathbf{a}}|} = \frac{\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}}{|\vec{\mathbf{a}}|^2}\vec{\mathbf{a}}$$

Cross Products **3D Only!!!!**

Given vectors $\vec{\mathbf{a}} = \langle a_1, a_2, a_3 \rangle$ and $\vec{\mathbf{b}} = \langle b_1, b_2, b_3 \rangle$, the cross product:

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \langle a_2 b_3 - a_3 b_2 \ , \ a_3 b_1 - a_1 b_3 \ , \ a_1 b_2 - a_2 b_1 \rangle$$

is *orthogonal* to both $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ and has length

$$|\vec{\mathbf{a}} \times \vec{\mathbf{b}}| = |\vec{\mathbf{a}}|\,|\vec{\mathbf{b}}| \sin\theta.$$

This is equal to the area of the parallelogram determined by $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.

Use the right hand rule to determine the direction of $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$.

- Use your *right hand* to swing from $\vec{\mathbf{a}}$ to $\vec{\mathbf{b}}$.
  Your thumb points in the direction of $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$.

(What do you get?)

Remembering $\langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$

Use the determinant of a 3×3 matrix.

$$\begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Example. Find $\langle 2, 3, 2 \rangle \times \langle 1, 0, 6 \rangle$, and show that it is $\perp$ to each.

Properties of $\times$

- $\vec{\mathbf{a}} \times \vec{\mathbf{a}} = \vec{\mathbf{0}}$
- $\vec{\mathbf{a}} \times \vec{\mathbf{b}} = -\vec{\mathbf{b}} \times \vec{\mathbf{a}}$
- $\vec{\mathbf{a}} \times (\vec{\mathbf{b}} + \vec{\mathbf{c}}) = \vec{\mathbf{a}} \times \vec{\mathbf{b}} + \vec{\mathbf{a}} \times \vec{\mathbf{c}}$
- $\vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) = (\vec{\mathbf{a}} \times \vec{\mathbf{b}}) \cdot \vec{\mathbf{c}}$
- $\vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) = (\vec{\mathbf{a}} \cdot \vec{\mathbf{c}})\vec{\mathbf{b}} - (\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})\vec{\mathbf{c}}$

Proofs by component manipulation

$$\vec{\mathbf{a}} \times (\vec{\mathbf{b}} + \vec{\mathbf{c}}) =$$
$$= \langle a_1, a_2, a_3 \rangle \times \big(\langle b_1, b_2, b_3 \rangle + \langle c_1, c_2, c_3 \rangle \big)$$
$$= \langle a_1, a_2, a_3 \rangle \times \langle b_1 + c_1, b_2 + c_2, b_3 + c_3 \rangle$$
$$\langle a_2(b_3+c_3) - a_3(b_2+c_2), a_3(b_1+c_1) - a_1(b_3+c_3), a_1(b_2+c_2) - a_2(b_1+c_1) \rangle$$
$$= \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle +$$
$$\langle a_2 c_3 - a_3 c_2, a_3 c_1 - a_1 c_3, a_1 c_2 - a_2 c_1 \rangle$$
$$= \vec{\mathbf{a}} \times \vec{\mathbf{b}} + \vec{\mathbf{a}} \times \vec{\mathbf{c}}$$

The quantity $|\vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} \times \vec{\mathbf{c}})|$ is called the **scalar triple product**, and calculates the volume of the *parallelepiped* determined by the vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, and $\vec{\mathbf{c}}$.

Physics

Application: Work

If a force applied in a direction (vector $\vec{\mathbf{F}}$)
causes a displacement in a direction (vector $\vec{\mathbf{D}}$),
then the work exerted is $W = \vec{\mathbf{F}} \cdot \vec{\mathbf{D}}$.

Application: Torque

If a force applied in a direction (vector $\vec{\mathbf{F}}$)
is applied to a lever, where the radius vector $\vec{\mathbf{r}}$ is
from the pivot to the place where the force is applied,
then a turning force called **torque** $\vec{\tau}$ is generated.
A formula is calculated by: $\vec{\tau} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}$