Lines, Planes, and Automobiles!

Lines in 2D Coordinates

Two common formats:

$y = mx + b$ (slope-intercept) or
$(y - y_0) = m(x - x_0)$ (pt-slope)

Given a point and a direction, you know the equation of the line.

$$\langle x, y \rangle = \langle x_0, y_0 \rangle + t\langle a, b \rangle$$

Two Lines

In three dimensions, two lines can

- be parallel
- intersect
- be skew

Lines in 3D Coordinates

$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$$

Each t gives a point (x, y, z) on L.
Reading componentwise, same as:

$$\left\{ \begin{aligned} x(t) &= x_0 + at \\ y(t) &= y_0 + bt \\ z(t) &= z_0 + ct \end{aligned} \right\}$$

Key idea: Read off direction vector $\vec{\mathbf{v}}$ from coeffs of t.

1D Examples

Example. Find the equation of the line that passes through $A = (2, 4, -3)$ and $B = (3, -1, 1)$.

Answer: To find the equation of a line, we need

- One Point.
- One Direction.

Example. Where does this line pass through the xy-plane?

Answer: In other words, ______________________________.

Never the twain shall meet

Example. Show that the following lines are skew.

$$\text{Romeo}: \langle 1+t, -2+3t, 4-t\rangle$$

$$\text{Juliet}: \langle 2s, 3+s, -3+4s\rangle$$

Answer: We will show:

- They are not **parallel**. (They would have the same ________.)
- They do not **intersect.** (There would be a point ________.)

Equations of planes

Question:
Does **a plane**
have a direction?

There is one vector __________ to the plane, the ______________ $\vec{\mathbf{n}}$.

Note: $\vec{\mathbf{n}}$ defines *infinitely many* planes. We also need a point.

A **plane** is defined by a normal vector $\vec{\mathbf{n}}$ and a point $\vec{\mathbf{r}}_0 = (x_0, y_0, z_0)$.
For any point $\vec{\mathbf{r}}$ on the plane, $\vec{\mathbf{r}} - \vec{\mathbf{r}}_0$ is perpendicular to $\vec{\mathbf{n}}$.
So the equation of a plane is

Alternate forms

$$\vec{\mathbf{n}} \cdot (\vec{\mathbf{r}} - \vec{\mathbf{r}}_0) = 0.$$

$$\langle a, b, c\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$ax + by + cz = d$$

Key idea: Read off normal vector $\vec{\mathbf{n}}$ from coeffs of x, y, z.

Plane Examples

Example. What is the angle between the planes

$$x + y + z = 1 \text{ and } x - 2y + 3z = 1?$$

Answer: When we need to find an angle, use ___________.

$$\theta = \cos^{-1}\left(\frac{2}{\sqrt{42}}\right) \approx 72^\circ$$

Example. What is the equation of the intersection line?

Answer: For the equation of a line, we need ___________________.

Plane Examples

Example. Find the distance from $(1, 0, -1)$ to $2x + 3y - 5z + 10 = 0$.
Answer: The normal vector to the plane is ____________,
so the shortest distance from $P_0 = (1, 0, -1)$ to the plane is along the line $(1, 0, -1) + t(2, 3, -5)$. Where does this hit the plane?

Use the equation of the line and the plane:

$$2x + 3y - 5z + 10 = 0 \rightsquigarrow 2(1 + 2t) + 3(0 + 3t) - 5(-1 - 5t) + 10 = 0$$

Simplifying, the point P_1 where the line hits the plane is when $t = \frac{-17}{38}$.

$$\vec{P}_1 - \vec{P}_0 = \tfrac{-17}{38}\langle 2, 3, -5\rangle, \text{ so}$$

$$|\vec{P}_1 - \vec{P}_0| = \tfrac{17}{38}\sqrt{2^2 + 3^2 + (-5)^2} = \tfrac{17}{\sqrt{38}}.$$