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Two Lines

In three dimensions, two lines can

- be parallel
- intersect
- be skew

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Answer: In other words, ________________________________.

Never the twain shall meet

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Example. What is the angle between the planes

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$2x + 3y - 5z + 10 = 0 \rightsquigarrow 2(1 + 2t) + 3(0 + 3t) - 5(-1 - 5t) + 10 = 0$

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$\vec{P}_1 - \vec{P}_0 = \frac{-17}{38}\langle 2, 3, -5\rangle$, so
$|\vec{P}_1 - \vec{P}_0| = \frac{17}{38}\sqrt{2^2 + 3^2 + (-5)^2} = \frac{17}{\sqrt{38}}$.