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Vector functions

$$\vec{r} : \mathbb{R} \rightarrow \mathbb{R}^3 \quad (\text{or } \mathbb{R}^2 \text{ or } \mathbb{R}^n)$$

$$\vec{r} : t \mapsto \langle f(t), g(t), h(t) \rangle$$

\vec{r} takes in a real number t
outputs a vector $\langle f(t), g(t), h(t) \rangle$

Limits and Helices

The **limit** of a vector function \vec{r} is defined by taking the limits of its component functions

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Plug in some values of t

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So $\vec{r} =$

Derivatives and Derivative-derivative definitions

Define $\vec{r}'(t) = \lim_{h \rightarrow 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h} = \langle f'(t), g'(t), h'(t) \rangle$.

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Example. $\int \langle 2 \cos t, \sin t, 2t \rangle dt = \langle 2 \sin t, -\cos t, t^2 \rangle + \vec{C}$.

Derivatives

Example. Find the equation of the tangent line to the helix $\vec{r}(t) = \langle 2 \cos t, \sin t, t \rangle$ at the point $P = (0, 1, \frac{\pi}{2})$.

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Game plan:

1. Find t^* for which the curve goes through the point P .
2. Find the tangent vector $\vec{r}'(t)$, plug in $t = t^*$.
3. Write the equation of the line.

Derivatives rule

- ▶ $\frac{d}{dt}(\vec{r}(t) + \vec{s}(t)) = \vec{r}'(t) + \vec{s}'(t)$
- ▶ $\frac{d}{dt}(c \vec{r}(t)) = c \vec{r}'(t)$
- ▶ $\frac{d}{dt}(f(t) \vec{r}(t)) = f'(t) \vec{r}(t) + f(t) \vec{r}'(t)$
- ▶ $\frac{d}{dt}(\vec{r}(t) \cdot \vec{s}(t)) = \vec{r}'(t) \cdot \vec{s}(t) + \vec{r}(t) \cdot \vec{s}'(t)$
- ▶ $\frac{d}{dt}(\vec{r}(t) \times \vec{s}(t)) = \vec{r}'(t) \times \vec{s}(t) + \vec{r}(t) \times \vec{s}'(t)$
- ▶ $\frac{d}{dt}(\vec{r}(f(t))) = f'(t) \vec{r}'(f(t))$

Motion in space

If $\vec{r}(t)$ is the vector position of a particle, then

- ▶ $\vec{r}'(t) = \vec{v}(t)$ is the vector velocity of the particle.
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Example. Show that if a particle moves with constant speed, then the velocity and acceleration vectors are orthogonal.