

Arc length

The arc length of a vector function is calculated by:

$$\int_{t=a}^{t=b} \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2} dt = \int_{t=a}^{t=b} |\vec{r}'(t)| dt$$

The **arc length function** is $s(t) = \int_{u=a}^{u=t} |\vec{r}'(u)| du$.

- ▶ We are using u as the parametrization variable instead of t .
- ▶ This is a function of t , telling how far along the curve you have traveled since a .

Example. Determine the distance that a particle travels from its initial position $(1, 0, 0)$ to any point on the curve

$$\vec{r}(t) = \cos t \vec{i} + \sin t \vec{j} + t \vec{k}.$$

Answer: We are looking for $s(t)$ starting at $t = \underline{\hspace{2cm}}$.

$$s(t) = \int_{u=0}^{u=t} \sqrt{\hspace{4cm}} du =$$

The distance travelled from time 0 to time t is $s(t) = \underline{\hspace{2cm}}$.

Reparametrization with respect to arc length

You may want to **reparametrize** your curve so that

one unit in your parameter \longleftrightarrow one unit in distance

To do this, we need to replace t by s .

Since we have s as a function of t , we need the **inverse function**!

In our example, $s = \sqrt{2}t$, so $t = \frac{s}{\sqrt{2}}$. Substituting,

$$\vec{r}(s) = \cos \frac{s}{\sqrt{2}} \vec{i} + \sin \frac{s}{\sqrt{2}} \vec{j} + \frac{s}{\sqrt{2}} \vec{k}.$$

Frenet Frame

There are many different parametrizations of any one curve.

The vectors $\vec{r}(t)$, $\vec{v}(t)$, $\vec{a}(t)$ depend on the parameter.

But the curve itself has intrinsic properties. **At every point:**

Three natural vectors make up the **Frenet frame**, or **TNB frame**.

\vec{T} The direction of the tangent vector. $\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$.

\vec{N} The direction in which the curve is turning. $\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$.

\vec{B} The third vector that completes \perp basis. $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$

A number that tells how bendy or twisty the curve is.

Definition: The **curvature** $\kappa(t)$ of a curve (“kappa”) tells how quickly \vec{T} is changing *with respect to distance traveled*.

$$\kappa = \left| \frac{d\vec{T}}{ds} \right| \underset{\text{chain rule}}{=} \left| \frac{\frac{d\vec{T}}{dt}}{\frac{ds}{dt}} \right| = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} \underset{\text{algebra}}{=} \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}.$$

The circle that lies along the curve has radius $1/\kappa$. (!)

Curvature

Example. Determine the vectors of the Frenet frame and the curvature of the curve $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$.

Frenet frame: We need $\vec{r}'(t) = \langle -\sin t, \cos t, 1 \rangle$ and $|\vec{r}'(t)| = \underline{\hspace{2cm}}$.

Then $\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \left\langle \frac{-\sin t}{\sqrt{2}}, \frac{\cos t}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$,

From this we find that $\vec{T}'(t) = \left\langle \frac{-\cos t}{\sqrt{2}}, \frac{-\sin t}{\sqrt{2}}, 0 \right\rangle$, so $|\vec{T}'(t)| = \underline{\hspace{2cm}}$.

Then $\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|} = \langle -\cos t, -\sin t, 0 \rangle$.

Now $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t) = \frac{1}{\sqrt{2}} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin t & \cos t & 1 \\ -\cos t & -\sin t & 0 \end{vmatrix} =$

The curvature $\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} =$

Question: Should $\kappa(t)$ be a constant?

Components of Acceleration

The curvature tells us about the centripetal force we feel.

Key idea: Understand \vec{a} in terms of the Frenet frame:

How much of the acceleration is toward \vec{T} , \vec{N} , and \vec{B} ?

Differentiate $\vec{v}(t) = v(t)\vec{T}(t)$. (magnitude (speed) times unit direction)

$$\vec{a} = v'\vec{T} + v\vec{T}' = v'\vec{T} + v(|\vec{T}'|\vec{N}) = v'\vec{T} + \kappa v^2\vec{N}.$$

► All acceleration is toward \vec{T} and \vec{N} . (Not to \vec{B} .)

a_T Toward \vec{T} : $a_T = v'$ is rate of change of speed.

a_N Toward \vec{N} : $a_N = \kappa v^2$. Curvature times speed squared!

Solve for a_T , a_N in terms of $\vec{r}(t)$.

$$\text{First, } \vec{v} \cdot \vec{a} = v\vec{T} \cdot (v'\vec{T} + \kappa v^2\vec{N}) = vv'\vec{T} \cdot \vec{T} + \kappa v^3\vec{T} \cdot \vec{N} = vv'$$

$$\text{So } a_T = v' = \frac{\vec{v} \cdot \vec{a}}{v} = \frac{\vec{r}'(t) \cdot \vec{r}''(t)}{|\vec{r}'(t)|} \text{ and } a_N = \kappa v^2 = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|}$$

Nice
symmetry!

Example. Find tang'l, normal comp's of acceleration for $\vec{r} = \langle t, 2t, t^2 \rangle$.