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No matter how close to $y = L$ you insist you must be (ε-close),

There is a way to choose a range δ around $x = a$ to ensure that

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More lines of thought

Example. Does the limit $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ exist?

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Consequence: If we know $f(x, y)$ is continuous at (a, b), then $\lim_{(x,y)\to(a,b)} f(x, y)$ exists!

Partial derivatives

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This is the **partial derivative of f with respect to x**. We write:

$$f_x(x, y) \quad \textbf{or} \quad \frac{\partial f}{\partial x} \quad \textbf{or} \quad \frac{\partial}{\partial x} f(x, y) \quad \textbf{or} \quad \frac{\partial z}{\partial x} \quad \textbf{or} \quad D_x f.$$

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★ **Idea:** Treat other variables as constants, differentiate normally. ★

Example. Let $f(x, y) = x^3 + x^2y^3 - 2y^2$. Find $f_x(2, 1)$ and $f_y(2, 1)$.

More examples

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$$\frac{\partial z}{\partial x} = \frac{-(3x^2 + 6yz)}{3z^2 + 6xy} \qquad \text{and} \qquad \frac{\partial z}{\partial y} = \frac{-(3y^2 + 6xz)}{3z^2 + 6xy}$$