More partials

This works with more variables too.

$\frac{\partial}{\partial z}\left(e^{xy} \ln z\right) =$ __________ and $\frac{\partial}{\partial x}\left(e^{xy} \ln z\right) =$ __________

We can also take higher derivatives.

$$\frac{\partial}{\partial x}\frac{\partial}{\partial x} f(x,y) \quad \textbf{or} \quad \frac{\partial^2}{\partial x^2} f(x,y) \quad \textbf{or} \quad f_{xx}(x,y)$$

We might even decide to mix our partial derivatives.

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\frac{\partial}{\partial x} f(x,y).$$

A big deal: Partial Differential Equations

- Laplace's Equation: $\frac{\partial^2}{\partial x^2} u(x,y) + \frac{\partial^2}{\partial y^2} u(x,y) = 0$ is a PDE.
  - Solutions (fcns u that satisfy) give formulas related to distribution of heat on a surface, how fluids & electricity flow.
- Wave Equation: $\frac{\partial^2}{\partial t^2} u(x,t) = a\frac{\partial^2}{\partial x^2} u(x,t)$ is a PDE.
  - Solutions describe the position of waves as a function of time.

Clairaut's Theorem

Example. Calculate all second-order partial derivatives of

$$f(x, y) = x^3 + x^2y^3 - 2y^2.$$

$f_x =$	$f_y =$
$f_{xx} =$	$f_{yx} =$
$f_{xy} =$	$f_{yy} =$

Notice: ______________________________

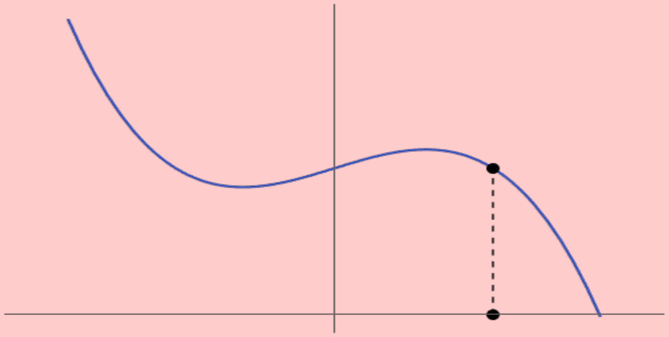
Clairaut's Theorem (mid 1700's)
Suppose $f(x, y)$ is defined on a disk D containing (a, b).
If f_{xy} and f_{yx} are continuous on D, **then** $f_{xy}(a, b) = f_{yx}(a, b)$.

Consequence: Order partial derivatives however you want.

$$f_{xyzz} = f_{zxyz} = f_{zyzx} = \cdots$$

Interpretation of partial derivatives

Function of one variable



$\frac{d}{dx} f(x)$ at $x = a$

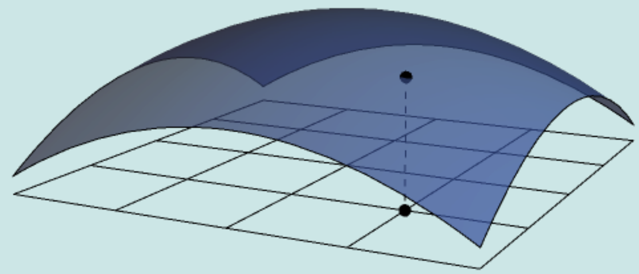
slope of tangent line to the curve

$$y = f(x)$$

at $x = a$.

"What is the rate of change of $f(x)$ as x changes?"

Function of several variables



$\frac{\partial}{\partial x} f(x, y)$ at $(x, y) = (a, b)$

slope of tangent line to the curve on the surface $z = f(x, y)$ where sliced by the vertical plane $y = b$ at $x = a$.

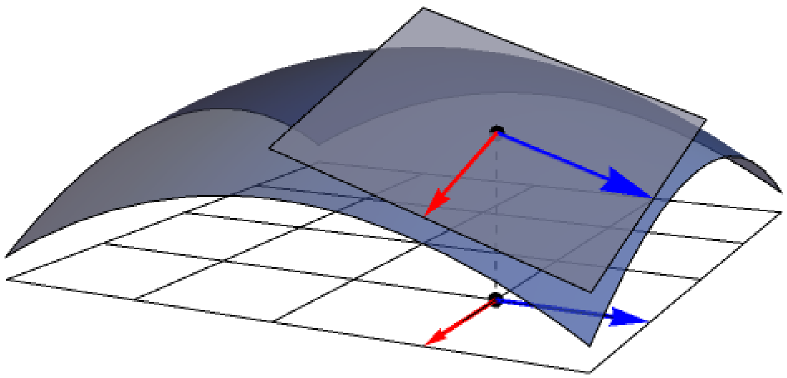
"If y is fixed, what is the rate of change of $f(x, y)$ as x changes?"

Tangenty

$\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at (x_0, y_0) are slopes of tangent lines along the surface.

They lie in the **tangent plane**.

"When we zoom into a smooth surface, the surface looks like a plane."



For any curve on the surface through $\left(x_0, y_0, f(x_0, y_0)\right)$, the tangent line to the curve would be in this plane too.

Key Idea: When the tangent plane to the surface is a good approximation for the the graph when (x, y) is near (a, b), we say that f is **differentiable** at (a, b).

Theorem. If the partial derivatives f_x and f_y **exist** near (a, b) and **are continuous** at (a, b), then f is differentiable at (a, b).

Planey

The equation of this tangent plane is easy. (point-slope)

$$(z - z_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Example. What is the eqn of the tangent plane to $z = xe^{xy}$ at $(1, 0)$?

Game Plan:

1. Find the partial derivatives, evaluate at $(1, 0)$. (Find slopes)

2. Determine the point on the surface. (Find point)

3. Write down equation of plane. (Point-slope formula)

Linear Approximation

Key idea: Use tangent plane as linear approximation to the function.

$T(x,y)$ gives a "good enough" value for $f(x,y)$ near (x_0, y_0).

Example. Use a linear approximation of $f(x,y) = xe^{xy}$ to approximate $f(1.1, -0.1)$.

Answer: $(1.1, -0.1)$ is a point near _________.

The tangent plane there is $T(x,y) = x + y$.

So: $f(1.1, -0.1) \approx T(1.1, -0.1) =$ ____________.

Note: $f(1.1, -0.1) = 1.1e^{-0.11} \approx .985$.

In more dimensions: Suppose $f(\vec{\mathbf{v}}) = f(w, x, y, z)$.

A linear approximation near $\vec{\mathbf{v}}_0 = (w_0, x_0, y_0, z_0)$ would be

$$f(\vec{\mathbf{v}}) - f(\vec{\mathbf{v}}_0) \approx f_w(\vec{\mathbf{v}}_0)(w - w_0) + f_x(\vec{\mathbf{v}}_0)(x - x_0) + f_y(\vec{\mathbf{v}}_0)(y - y_0) + f_z(\vec{\mathbf{v}}_0)(z - z_0)$$

Differentials

Differentials are the other way to understand linear approximations. How much does z change as x and y change?

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy$$

dz is a *approximation* for how much z actually changes.

Example. If $z = x^2 + 3xy - y^2$, find dz.

$dz =$

Conclusion: If x changes from $2 \to 2.05$, $dx =$ ______

If y changes from $3 \to 2.96$, $dy =$ ______.

We would expect z to change by __________.

The true change is .6449.