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So $z = f(g(t), h(t))$.

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Key idea:

You must add contributions from all dependencies.

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Example. Let $z = x^2y + 3xy^2$, where $x = \sin 2t$, $y = \cos t$. Find $\frac{dz}{dt}$, $z'(0)$.

Answer:

More Chains

Alternatively, we might have $z = f(x, y)$

and $x = g(s, t)$, $y = h(s, t)$.

Then $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}$.

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Example. Consider $u = x^4y + y^2z^3$ where $x = rse^t$, $y = rs^2e^{-t}$, $z = r^2s(\sin t)$. Find $\frac{\partial u}{\partial s}$.

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In full generality: If u is a function of x_1, x_2, \dots, x_n
and each x_j is a function of t_1, t_2, \dots, t_m , then

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2} \cdot \frac{\partial x_2}{\partial t_i} + \dots + \frac{\partial u}{\partial x_n} \cdot \frac{\partial x_n}{\partial t_i}.$$

Implicit differentiation

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Example. Find $\frac{\partial z}{\partial x}$ if $x^3 + y^3 + z^3 + 6xyz = 1$