# Local Extrema

## Functions of one variable

$f(x)$ has a **local maximum** at $x=a$ if for all points $x$ near $a$, $f(x) \leq f(a)$.
$f(x)$ has a **local minimum** at $x=a$ if for all points $x$ near $a$, $f(x) \geq f(a)$.

(What about global/absolute?)

**If** $f(x)$ has a local max or local min at $x=a$, **then:**

$f'(a)=0$ or $f'(a)$ does not exist.

A point $a$ where this is true is called a **critical point**.

*However,* **If** $f'(a)=0$ or $f'(a)$ DNE **then** this **does not imply** that $x=a$ is a local max or min.

## Functions of multiple variables

$f(x, y)$ has a **local maximum** at $(a, b)$ if for all points nearby, $f(x, y) \leq f(a, b)$.
$f(x, y)$ has a **local minimum** at $(a, b)$ if for all points nearby, $f(x, y) \geq f(a, b)$.

(What about global/absolute?)

**If** $f(x, y)$ has a local max or local min at $(x, y)=(a, b)$, **then:**

$f_x(a, b)=0$ and $f_y(a, b)=0$ (or DNE)

A point $(a, b)$ where this is true is called a **critical point**.

*However,* **If** $(f_x$ and $f_y)=0$ or DNE **then** this **does not imply** that $(x, y)=(a, b)$ is a local max or min.

# Determining local extrema

Important vocabulary:

- A maximum or minimum: means ________________.
- A maximum value or minimum value: means ________.

We can try to determine if a critical point is a local extremum using:

**The second derivative test.**

**If** the second partial derivatives of $f(x, y)$ are continuous around $(a, b)$
**And if** $f_x(a, b) = 0$ and $f_y(a, b) = 0$, **then** define $D(a, b)$:

$$D(a, b) = f_{xx}(a, b) \cdot f_{yy}(a, b) - \left(f_{xy}(a, b)\right)^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$

1. If $D > 0$ and $f_{xx} > 0$, then $(a, b)$ is a local minimum.
2. If $D > 0$ and $f_{xx} < 0$, then $(a, b)$ is a local maximum.
3. If $D < 0$, then $(a, b)$ is a **saddle point** of $f$.
4. If $D = 0$, the test is inconclusive.

# Extreme Examples

Example. Find the local extrema and saddle points of

$$f(x, y) = x^4 + y^4 - 4xy + 1.$$

- Critical points:

- For each: Find $D(a, b)$, classify.

# Absolute (global) Extrema

| Functions of one variable | Functions of multiple variables |
| --- | --- |
| **Extreme Value Theorem:** If $f$ is continuous on a **closed interval**, then $f$ attains an absolute max and absolute min **somewhere** on this interval. | **Extreme Value Theorem:** If $f$ is continuous on a __________ **set** in $\mathbb{R}^2$, then $f$ attains an absolute max and absolute min **somewhere** on this set. |

**Strategy for absolute extrema:**

Check for interior critical points

- Find all critical points.
- Which are inside the set?
- Evaluate function there.

Find boundary extreme values

- Parametrize the boundary. pieces? (Including endpoints!) 1-Var fcn
- Find critical points of 1-Var fcn.
- Evaluate function there & endpts.

Largest, smallest function values determine absolute extrema on set.

# Optimization is just finding maxima and minima

Example. Find the global extrema of $f(x,y) = x^2 - 2xy + 2y$ on the rectangle $0 \le x \le 3$ and $0 \le y \le 2$.

- Draw a picture of set.
- Check for critical points on interior.

$$\left.\begin{array}{r} 0 = f_x = 2x - 2y \\ 0 = f_y = -2x + 2 \end{array}\right\} \to \left\{\begin{array}{c} x = y \\ x = 1 \end{array}\right\} \to \begin{array}{l} (1,1) \text{ is crit. pt.} \\ f(1,1) = \_\_\_\_\_\_ \end{array}$$

- Find extreme values along boundary. (Piecewise-defined!)

$f(x,0) =$ __________ $\rightsquigarrow$ min: __________ max: __________

$f(3,y) = 9 - 4y$ $\rightsquigarrow$ min: 1 @ (3,2) max: 9 @ (3,0)

$f(x,2) = x^2 - 4y + 4$ $\rightsquigarrow$ min: __________ max: __________

$f(0,y) = 2y$ $\rightsquigarrow$ min: 0 @ (0,0) max: 4 @ (0,2)

- Determine largest & smallest.

# Optimization is just finding maxima and minima

Example. A rectangular box with no lid is made from 12 $\text{m}^2$ of cardboard. What is the maximum volume of the box?

*Solution.* Let length, width, and height be $x$, $y$, and $z$, respectively. Then the question asks us to maximize $V =$ __________,
subject to ________________.

Solving for $z$ gives $z = \frac{12-xy}{2x+2y}$. Inserting, $V = xy\left(\frac{12-xy}{2x+2y}\right)$.

To find an optimum value, solve for $\frac{\partial V}{\partial x} = 0$ and $\frac{\partial V}{\partial x} = 0$.
$\frac{\partial V}{\partial x} = 0 \rightsquigarrow$
$\frac{\partial V}{\partial y} = 0 \rightsquigarrow$

Solving these simultaneous equations, $12 - 2xy = x^2 = y^2 \Rightarrow x = \pm y$.
Because this is real world, ______, so we solve $12 - 3x^2 = 0$: ______.

This problem must have an absolute maximum, which must occur at a critical point. (Why?) Therefore $(x, y, z) = (2, 2, 1)$ is the absolute maximum, and the maximum volume is $xyz = 4$.