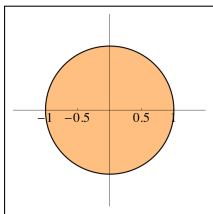
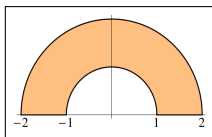


Some regions *demand* polar coordinates

Some regions are best described in polar coordinates:



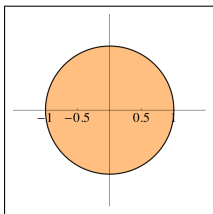
Parts of circles



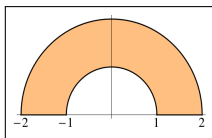
Parts of annuli

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Parts of circles



Parts of annuli

These are called **polar rectangles** because they have the form

$$\left\{ \begin{array}{l} a \leq r \leq b \\ \alpha \leq \theta \leq \beta \end{array} \right\} \text{ for constants } \left\{ \begin{array}{l} a, b \\ \alpha, \beta \end{array} \right\}$$

Polar coordinates

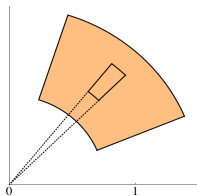
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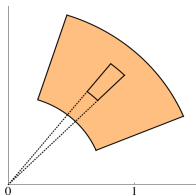
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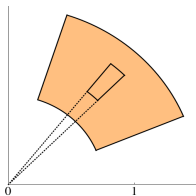
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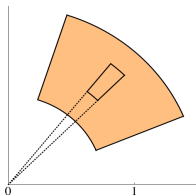
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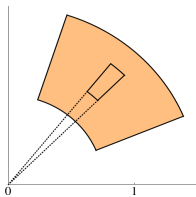
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(How to remember?)

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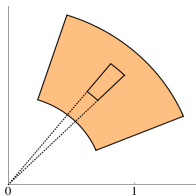
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Consequence: To calculate $\iint_D f(x, y) dA$, when D is best described in polar coordinates, calculate

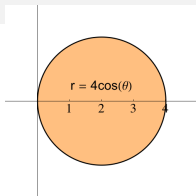
$$\iint_D f(r \cos \theta, r \sin \theta) r dr d\theta$$

Polar coordinate example

Example. Find the area inside
 $r = 4 \cos \theta$ from $\theta = \frac{\pi}{4}$ to $\frac{\pi}{2}$.

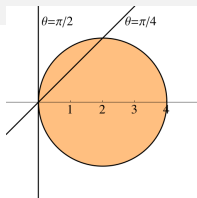
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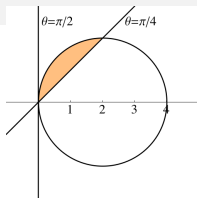
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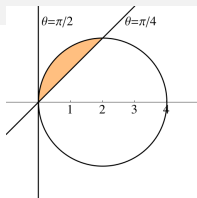
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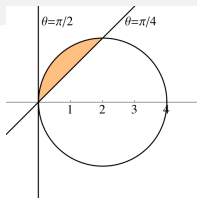


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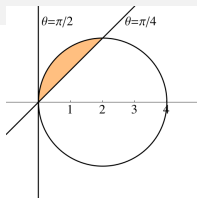


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★ Along the way, we had $\int_{\alpha}^{\beta} \frac{r^2}{2} d\theta \dots\dots\dots$

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$$\dots = \int_{\theta=0}^{\theta=\pi} K \frac{a^3}{3} d\theta = \left[K \frac{a^3}{3} \theta \right]_{\theta=0}^{\theta=\pi} = \frac{\pi K a^3}{3}.$$

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