Course Notes

Multivariable Calculus, Fall 2015

Queens College, Math 201

Prof. Christopher Hanusa

`http://qc.edu/~chanusa/courses/201/15/`

Class Introductions

Arrange yourselves into groups of four or five people,
With people you **don't know**.

- Introduce yourself. (your name, where you're from, your major)
- What brought you to this class?
- Fill out **the blank side of** your notecard:
  - Write your name. (Stylize if you wish.)
  - Write some words about how I might remember you & your name.
  - *Draw* something (anything!) in the remaining space.
- *Small talk suggestion:* What kept you busy this summer?
- Exchange contact information. (phone / email / other)

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  - Organize into themes.
- How do these ideas translate to *multivariable* calculus?

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- **Form good study groups !!!!!!!!**
  - Discuss classwork and homework. Study for exams.
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All homeworks posted online; first one (many parts) due Tuesday.

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There are two types of homework in this class:

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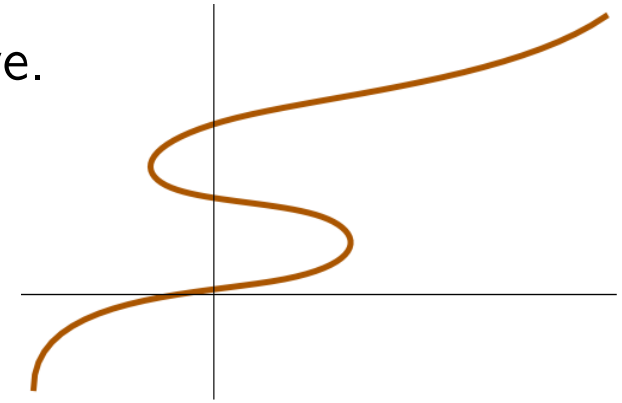
Parametric Curves

Imagine a particle traveling along this curve.



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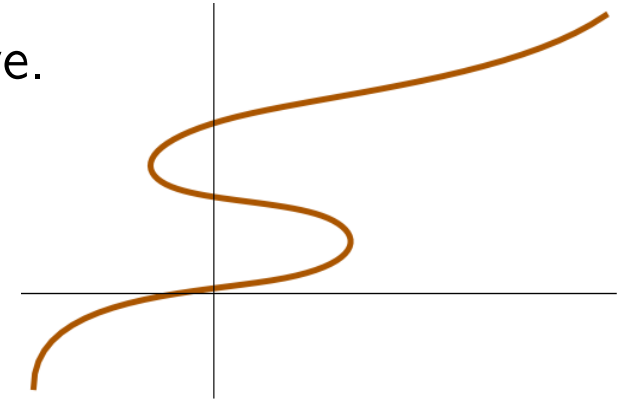

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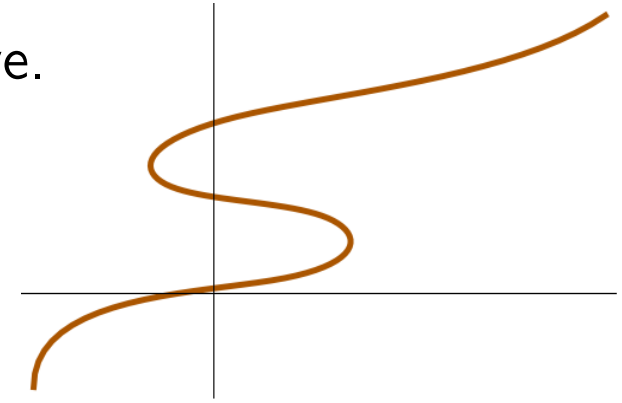

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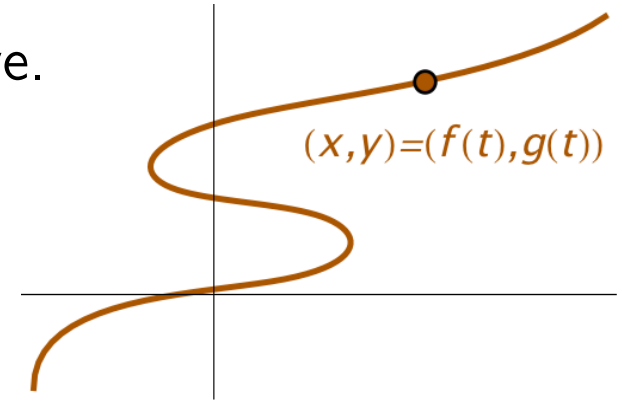

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the x-coordinate and the y-coordinate
of the particle as a function of "time".

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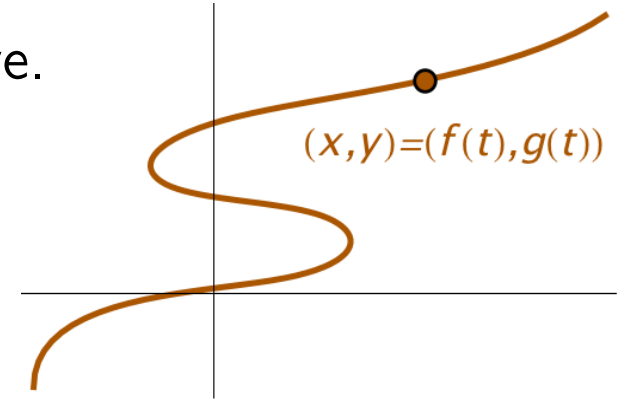

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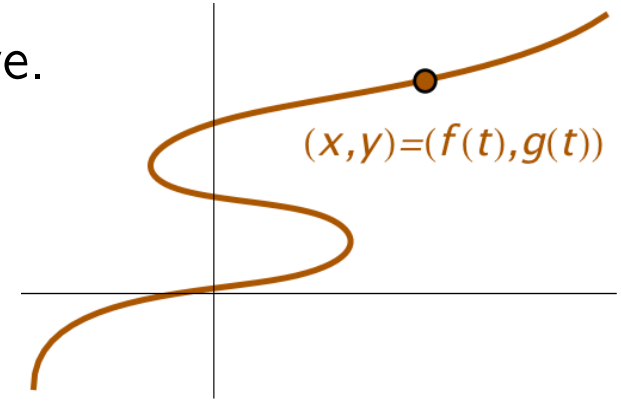

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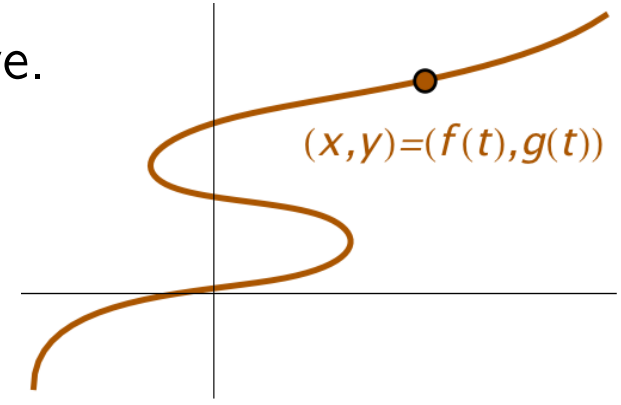

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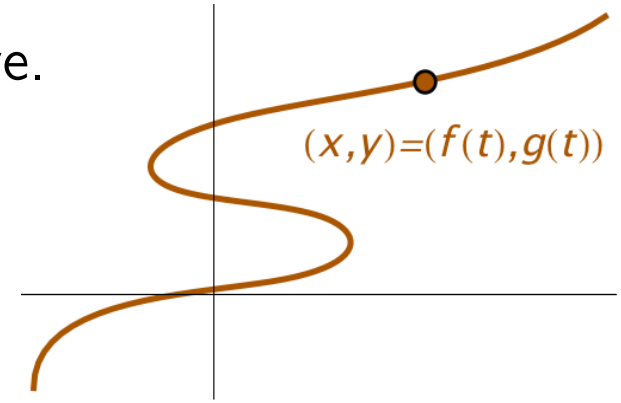

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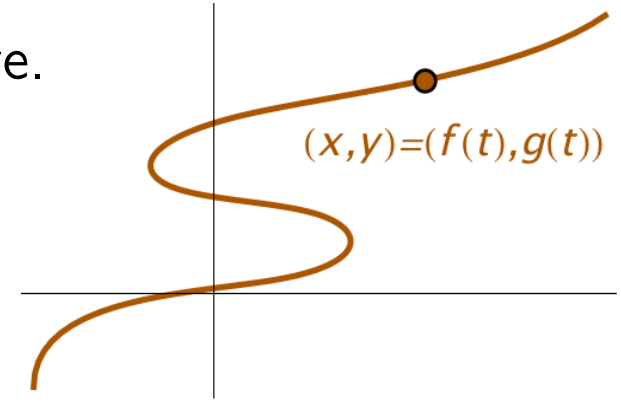

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Goal 1: *Understand* parametric curves. (Today)

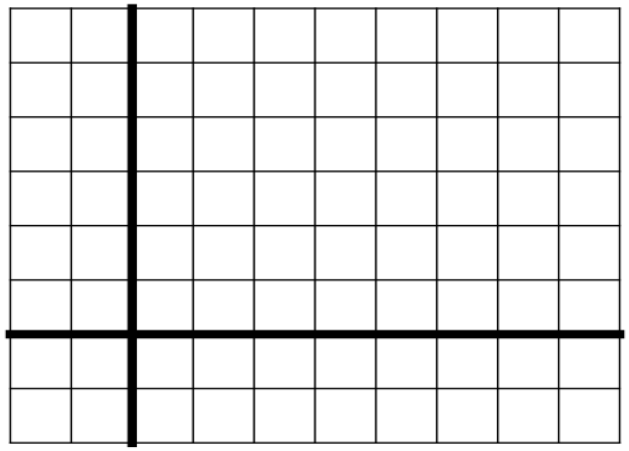
Goal 2: *Do calculus* using parametric curves. (Next time)

Sketching Parametric Curves

What is the shape of a curve given by parametric equations?

- By hand
- Use a calculator or computer

Example. Plot the curve defined by $x(t) = t^2 - 2t$ and $y(t) = t + 1$.



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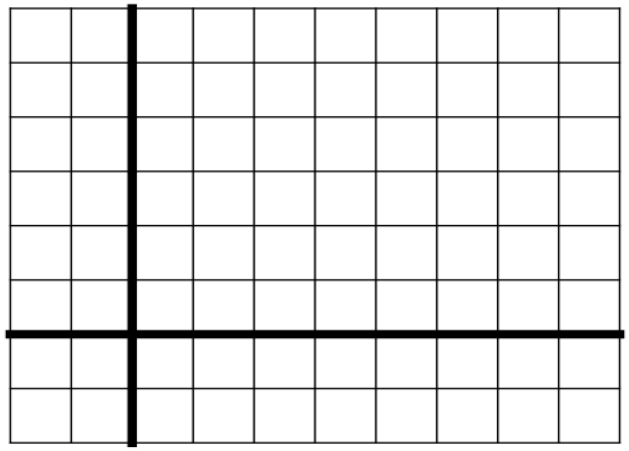

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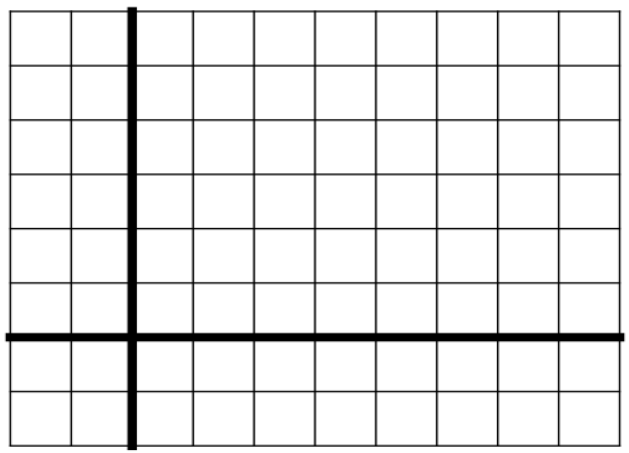

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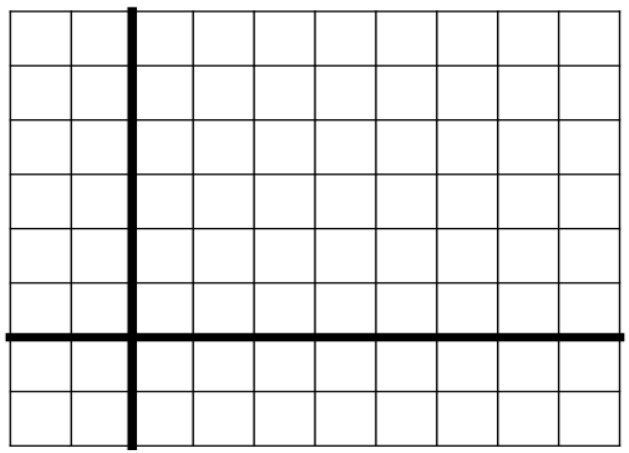

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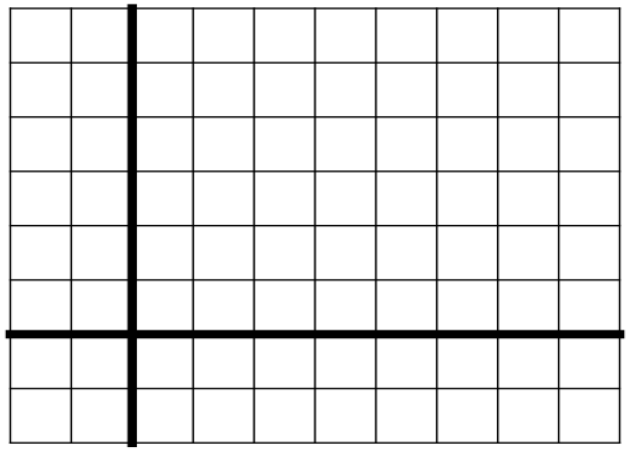

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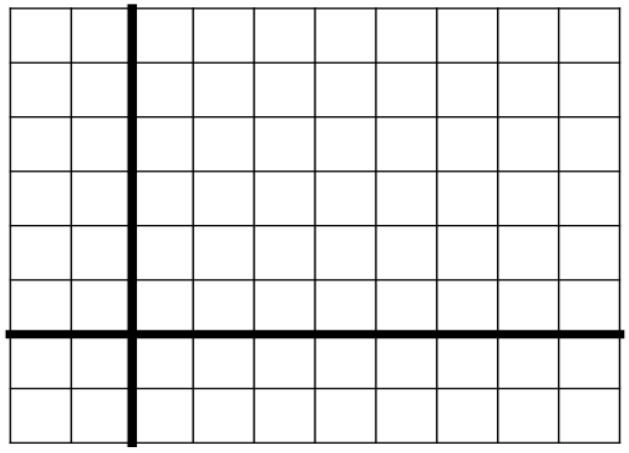

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Solve for t in second equation: $t = y - 1$ and plug in:
$x = (y-1)^2 - 2(y-1) = y^2 - 4y + 3$, a "sideways parabola".

Around and Around

Example. Plot the curve defined by $x = \cos t$, $y = \sin t$, $0 \leq t \leq 2\pi$.

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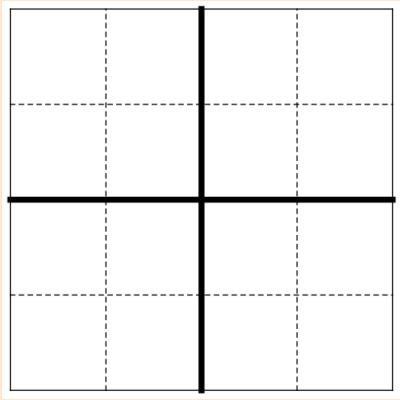

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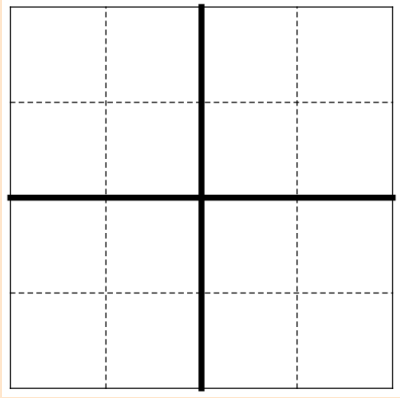

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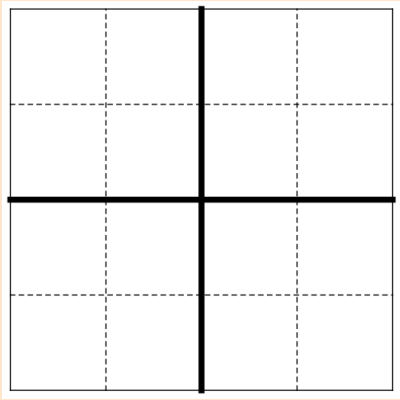

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Example. Is this the same as $x = \cos 2t$, $y = \sin 2t$, $0 \le t \le 2\pi$?

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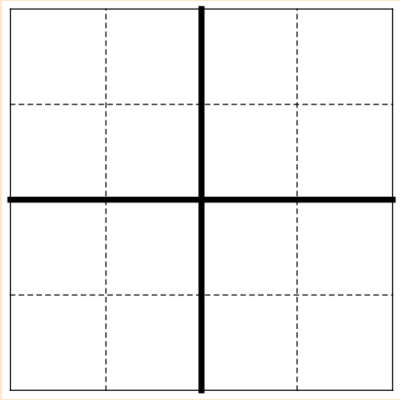

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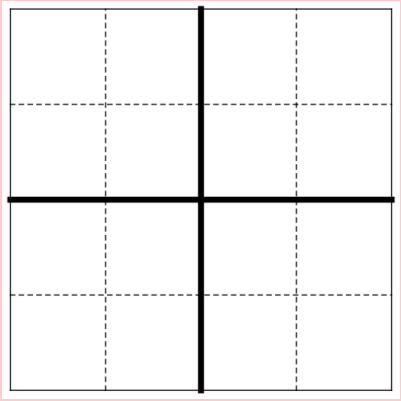

Question: What is $x^2 + y^2$?

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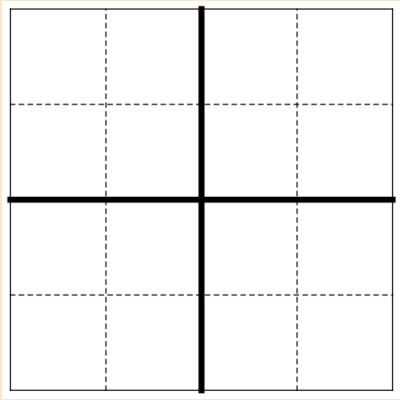

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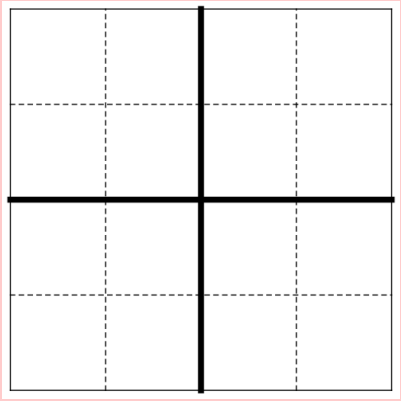

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The figures traced out (the curves) are the same but the functions **are not** the same.

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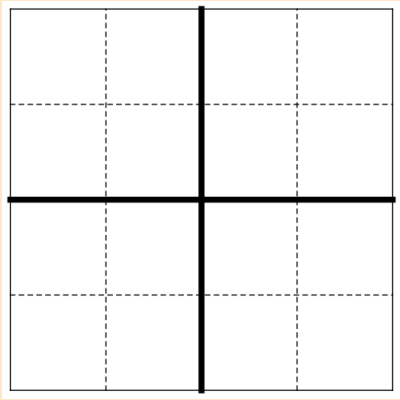
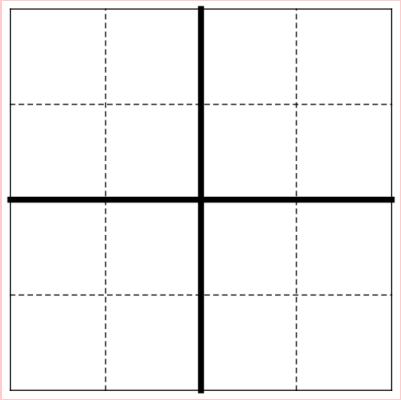


A circle! But you knew that.
$\cos^2 t + \sin^2 t = 1$

Starts at $t = 0$: $(1, 0)$ and goes around counterclockwise.

Example. Is this the same as $x = \cos 2t$, $y = \sin 2t$, $0 \le t \le 2\pi$?



Question: What is $x^2 + y^2$?

The figures traced out (the curves) are the same but the functions **are not** the same.

You need to know your trig functions and values at certain points!!!!

Circumnavigation

If we want to draw a circle at some other place

$$(x-h)^2+(y-k)^2=r^2,$$

set $x-h=r\cos t$ and $y-k=r\sin t$.

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Try it out! Get out your graphing calculator TI-(≤ 86).

Switch to Parametric mode: `MODE` ↓ ↓ ↓ `PAR` (`Enter`).

Set the domain of T to be from 0 to 2π.
`WINDOW: Tmin = 0`, `Tmax = 2π`, `Tstep = π/10`.

Enter the equations X_1 `= 3 cos(T) + 2` and Y_1 `= 3 sin(T) + 4`.

This plots a circle of radius 3 centered at $(2,4)$.

Computers to the rescue

Calculators and computers can graph much more complicated curves.

$$x_1(t) = t + 2\sin(2t) \quad \text{and} \quad y_1(t) = t + 2\cos(5t)$$

$$x_2(t) = 1.5\cos t - \cos 30t \quad \text{and} \quad y_2(t) = 1.5\sin t - \sin 30t$$

$$x_3(t) = \sin(t + \cos 100t) \quad \text{and} \quad y_3(t) = \cos(t + \sin 100t)$$

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Tools:

- Wolfram Alpha `http://www.wolframalpha.com/`
- More powerful is Wolfram *Mathematica*. Get license from MyQC:
  `myqc.qc.cuny.edu/Academics/mathematics/Pages3/access.aspx`
- Online plotter: `desmos.com` Put $(f(t), g(t))$ in parentheses.
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Next time: What is the shape of a parametric curve? What is the length of a parametric curve? What about polar coordinates?

Before then: Work on homework to present in class Wednesday. Email me contact info, do syllabus quiz. Play with parametric eqns.