Area under a parametric curve

Given $y = f(x)$, the area under the curve from $x = a$ to $x = b$ is

$$\text{Area} = \int_{x=a \ \leftarrow \text{ left endpoint}}^{x=b \ \leftarrow \text{ right endpoint}} y\,dx \qquad = \int_{t=\alpha \ \leftarrow \text{ left endpoint}}^{t=\beta \ \leftarrow \text{ right endpoint}} g(t)\,f'(t)\,dt$$

Example. Find the area under one arch of the cycloid $\begin{cases} x = r(\theta - \sin\theta) \\ y = r(1 - cos\theta) \end{cases}$
(Here, r is a constant and θ is the parameter.)

Plot it to see the shape. One arch has range ___ $\leq \theta \leq$ ____.

$$A = \int y\,dx = \int r(1 - \cos\theta)r(1 - \cos\theta)\,d\theta$$
$$= r^2 \int (1 - 2\cos\theta + \cos^2\theta)\,d\theta$$

$$= 3\pi r^2$$

Area inside a polar curve

For cartesian functions $y = f(x)$, calculate area as $A = \int dA = \int y\, dx$.

What about the area "*inside a curve*" best described as a polar function $r = f(\theta)$?

- We still use $A = \int dA$.
- But the formula for dA is different.

$$A = \int_{\theta=a}^{\theta=b} \frac{1}{2}\left[f(\theta)\right]^2 d\theta = \int_a^b \frac{1}{2} r^2\, d\theta$$

Polar land

How much area is swept out by a little slice?

Example. What is the area inside one loop of the four-leaved rose $r = \cos 2\theta$?

- What are the bounds on θ?
- For which θ does the curve pass through the origin?

$$A = \int_{\theta=\_\_}^{\theta=\_\_} \frac{1}{2}\left[\_\_\_\_\_\_\right]^2 d\theta$$

Inside yet Outside

Example. Calculate the area inside the curve $r = 3\sin\theta$ and outside the curve $r = \sin\theta + 1$.

First: Draw a picture!

- What are these curves?
- Where do they intersect?

Now calculate: $$\int_{\theta=\_\_\_\_}^{\theta=\_\_\_\_} \Big[\underbrace{\left(\tfrac{1}{2}(3\sin\theta)^2\right)}_{\textit{outside}} - \underbrace{\left(\tfrac{1}{2}(\sin\theta+1)^2\right)}_{\textit{inside}} \Big]\, d\theta$$

$$= \frac{1}{2}\int_{\theta=\_\_}^{\theta=\_\_} \left[9\sin^2\theta - (\sin^2\theta + 2\sin\theta + 1)\right] d\theta$$

$$= \frac{1}{2}\int_{\theta=\_\_}^{\theta=\_\_} \left[8\sin^2\theta - 2\sin\theta - 1\right] d\theta$$

$$= \frac{1}{2}\int_{\theta=\_\_}^{\theta=\_\_} \left[4 - 4\cos 2\theta - 2\sin\theta - 1\right] d\theta$$

$$= \frac{1}{2}\int_{\theta=\_\_}^{\theta=\_\_} \left[3 - 4\cos 2\theta - 2\sin\theta\right] d\theta$$

$$= \left[\frac{3\theta}{2} - \sin 2\theta + \cos\theta\right]_{\pi/6}^{5\pi/6}$$

$$= \frac{3}{2}\left(\frac{5\pi}{6} - \frac{\pi}{6}\right) - \left(\sin\frac{10\pi}{6} - \sin\frac{2\pi}{6}\right) + \left(\cos\frac{5\pi}{6} - \cos\frac{\pi}{6}\right) = \cdots$$

$$= \pi + \sqrt{3} - \sqrt{3} = \pi$$

Arc length of a parametric curve

To find the arc length of a parametric curve, think $L = \int dL$.

How much arc length ΔL does the curve traverse in one time unit Δt?

$$\Delta L_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{(f'(t_i^*)\Delta t_i)^2 + (g'(t_i^{**})\Delta t_i)^2}$$

(there is some t_i^* and some t_i^{**} in the time interval...)

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta L_i = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{(f'(t_i^*))^2 + (g'(t_i^{**}))^2}\Delta t_i$$

(Similar to a Riemann Sum)

$$L = \int_{t=\alpha}^{t=\beta} \sqrt{\left[f'(t)\right]^2 + \left[g'(t)\right]^2}\, dt$$

Example. Find the arc length for the parametric curve $\begin{cases} x = \sin 2t \\ y = \cos 2t \end{cases}$ for $0 \le t \le 2\pi$. What do we expect? What is this curve?

$\int_{t=0}^{t=2\pi} \sqrt{\left[f'(t)\right]^2 + \left[g'(t)\right]^2}\, dt =$

Arc length of a polar curve

To calculate arc length, view a polar curve as a parametric curve.

- Convert $r = f(\theta)$ to $\begin{cases} x = f(\theta)\cos\theta \\ y = f(\theta)\sin\theta \end{cases}$
- Then $L = \displaystyle\int_{\theta=\alpha}^{\theta=\beta} \sqrt{\left[\frac{d}{d\theta}\big(f(\theta)\cos\theta\big)\right]^2 + \left[\frac{d}{d\theta}\big(f(\theta)\sin\theta\big)\right]^2}\, d\theta$

. . . take derivatives & do the algebra . . .

$$L = \int_{\theta=\alpha}^{\theta=\beta} \sqrt{\big[f(\theta)\big]^2 + \left[\frac{d}{d\theta}\big(f(\theta)\big)\right]^2}\, d\theta$$

$$= \int_{\theta=\alpha}^{\theta=\beta} \sqrt{r^2 + \left(\tfrac{dr}{d\theta}\right)^2}\, d\theta$$

We can also understand this as $dL = \sqrt{(r\,d\theta)^2 + (dr)^2}$