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... take derivatives & do the algebra ...

$$L = \int_{\theta=\alpha}^{\theta=\beta} \sqrt{\big[f(\theta)\big]^2 + \left[\frac{d}{d\theta}\big(f(\theta)\big)\right]^2}\, d\theta$$

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To calculate arc length, view a polar curve as a parametric curve.

- Convert $r = f(\theta)$ to $\begin{cases} x = f(\theta)\cos\theta \\ y = f(\theta)\sin\theta \end{cases}$
- Then $L = \displaystyle\int_{\theta=\alpha}^{\theta=\beta} \sqrt{\left[\frac{d}{d\theta}\big(f(\theta)\cos\theta\big)\right]^2 + \left[\frac{d}{d\theta}\big(f(\theta)\sin\theta\big)\right]^2}\, d\theta$

... take derivatives & do the algebra ...

$$L = \int_{\theta=\alpha}^{\theta=\beta} \sqrt{\big[f(\theta)\big]^2 + \left[\frac{d}{d\theta}\big(f(\theta)\big)\right]^2}\, d\theta$$
$$= \int_{\theta=\alpha}^{\theta=\beta} \sqrt{r^2 + \left(\tfrac{dr}{d\theta}\right)^2}\, d\theta$$

We can also understand this as $dL = \sqrt{(r\, d\theta)^2 + (dr)^2}$