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Key idea: Use the dot product to find the angle between vectors.

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Example. What is the angle between $\vec{\mathbf{a}} = \langle 2, 2, -1 \rangle$ and $\vec{\mathbf{b}} = \langle 5, -3, 2 \rangle$?
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Question: What happens when two vectors are orthogonal?

Key idea: Two vectors are orthogonal if and only if ____________.

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Dot products let you project one vector onto another.

Answers: "How far does vector $\vec{\mathbf{b}}$ go in vector $\vec{\mathbf{a}}$'s direction?"

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The unit vector in $\vec{\mathbf{a}}$'s direction is ________.

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Therefore

$$\text{proj}_{\vec{\mathbf{a}}}\vec{\mathbf{b}} = \frac{\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}}{|\vec{\mathbf{a}}|}\cdot\frac{\vec{\mathbf{a}}}{|\vec{\mathbf{a}}|} = \frac{\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}}{|\vec{\mathbf{a}}|^2}\vec{\mathbf{a}}$$

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(What do you get?)

Remembering $\langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$

Use the determinant of a 3×3 matrix.

$$\begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Example. Find $\langle 2, 3, 2 \rangle \times \langle 1, 0, 6 \rangle$, and show that it is $\perp$ to each.

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$\vec{\mathbf{a}} \times (\vec{\mathbf{b}} + \vec{\mathbf{c}}) =$

$= \langle a_1, a_2, a_3 \rangle \times \big(\langle b_1, b_2, b_3 \rangle + \langle c_1, c_2, c_3 \rangle\big)$

$= \langle a_1, a_2, a_3 \rangle \times \langle b_1 + c_1, b_2 + c_2, b_3 + c_3 \rangle$

$\langle a_2(b_3+c_3) - a_3(b_2+c_2), a_3(b_1+c_1) - a_1(b_3+c_3), a_1(b_2+c_2) - a_2(b_1+c_1) \rangle$

$= \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle +$

Properties of $\times$

- $\vec{\mathbf{a}} \times \vec{\mathbf{a}} = \vec{\mathbf{0}}$
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$$= \vec{\mathbf{a}} \times \vec{\mathbf{b}} + \vec{\mathbf{a}} \times \vec{\mathbf{c}}$$

The quantity $|\vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} \times \vec{\mathbf{c}})|$ is called the **scalar triple product**, and calculates the volume of the *parallelepiped* determined by the vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, and $\vec{\mathbf{c}}$.

Physics

Application: Work

If a force applied in a direction (vector $\vec{\mathbf{F}}$) causes a displacement in a direction (vector $\vec{\mathbf{D}}$), then the work exerted is $W = \vec{\mathbf{F}} \cdot \vec{\mathbf{D}}$.

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Application: Torque

If a force applied in a direction (vector $\vec{\mathbf{F}}$)
is applied to a lever, where the radius vector $\vec{\mathbf{r}}$ is
from the pivot to the place where the force is applied,
then a turning force called **torque** $\vec{\tau}$ is generated.
A formula is calculated by: $\vec{\tau} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}$