Functions

Single-variable functions

$f : \mathbb{R} \to \mathbb{R}$

$f : x \mapsto f(x)$

f takes in a real number x

outputs a real number $f(x)$

Vector functions

$\vec{\mathbf{r}} : \mathbb{R} \to \mathbb{R}^3$ (or $\mathbb{R}^2$ or $\mathbb{R}^n$)

$\vec{\mathbf{r}} : t \mapsto \langle f(t), g(t), h(t) \rangle$

$\vec{\mathbf{r}}$ takes in a real number t

outputs a vector $\langle f(t), g(t), h(t) \rangle$

Limities and Helices

The **limit** of a vector function $\vec{\mathbf{r}}$ is defined by taking the limits of its component functions (as long as each of these exists...)

$$\lim_{t\to a} \vec{\mathbf{r}}(t) = \left\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \right\rangle$$

A vector-valued function $\vec{\mathbf{r}}(t)$ is continuous at a if ________________.

Example. Sketch the curve given by $\vec{\mathbf{r}}(t) = \cos t\,\vec{\mathbf{i}} + \sin t\,\vec{\mathbf{j}} + t\,\vec{\mathbf{k}}$.
The x and y components __________ while the z component _____.
Plug in some values of t

Intersectionnnnnnnnnn

Example. Find a vector function that is the intersection of the cylinder $x^2 + z^2 = 1$ and the plane $y + z = 2$.

Strategems: Find a parametrization... What parameter to use?

- If a curve is oriented in one direction, use that variable as t.
- When the curve is closed, this is not possible—work first in 2D.

Answer: Use the fact that we are on the cylinder. (Eqn 1) Project onto the xz-plane and start the parametrization there:

$x(t) =$ $z(t) =$ $\_\_\_\_ \leq t \leq \_\_\_\_.$

Use (Eqn 2) to find the y-coordinate:

So $\vec{\mathbf{r}} =$

Derivatives and Derivative-derivative definitions

Define $\vec{\mathbf{r}}'(t) = \lim\limits_{h\to 0} \frac{\vec{\mathbf{r}}(t+h)-\vec{\mathbf{r}}(t)}{h} = \langle f'(t), g'(t), h'(t)\rangle$.
The **derivative** $\vec{\mathbf{r}}'(t)$ of a vector-valued function is a vector in the direction tangent to the curve $\vec{\mathbf{r}}(t)$.

- Standardize. The **unit tangent vector** $\vec{\mathbf{T}} = \frac{\vec{\mathbf{r}}'(t)}{|\vec{\mathbf{r}}'(t)|}$.
- We can take multiple derivatives $\vec{\mathbf{r}}''(t) = \frac{d}{dt}\big(\vec{\mathbf{r}}'(t)\big)$
- A function is **smooth** on an interval I if
  - $\vec{\mathbf{r}}'(t)$ is continuous on I
  - and $\vec{\mathbf{r}}'(t) \neq \vec{\mathbf{0}}$, except possibly at the endpoints of I

We can integrate too. $\int_a^b \vec{\mathbf{r}}(t)\,dt = \left\langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt\right\rangle$

Remember: Indefinite integrals have a (vector) constant of integration.
Example. $\int \langle 2\cos t, \sin t, 2t\rangle\,dt = \langle 2\sin t, -\cos t, t^2\rangle + \vec{\mathbf{C}}$.

Derivatives

Example. Find the equation of the tangent line to the helix $\vec{\mathbf{r}}(t) = \langle 2\cos t, \sin t, t\rangle$ at the point $P = (0, 1, \frac{\pi}{2})$.

Game plan:

1. Find t^* for which the curve goes through the point P.

2. Find the tangent vector $\vec{\mathbf{r}}'(t)$, plug in $t = t^*$.

3. Write the equation of the line.

Derivatives rule

- $\frac{d}{dt}\big(\vec{\mathbf{r}}(t) + \vec{\mathbf{s}}(t)\big) = \vec{\mathbf{r}}'(t) + \vec{\mathbf{s}}'(t)$
- $\frac{d}{dt}\big(c\,\vec{\mathbf{r}}(t)\big) = c\vec{\mathbf{r}}'(t)$
- $\frac{d}{dt}\big(f(t)\,\vec{\mathbf{r}}(t)\big) = f'(t)\,\vec{\mathbf{r}}(t) + f(t)\,\vec{\mathbf{r}}'(t)$
- $\frac{d}{dt}\big(\vec{\mathbf{r}}(t) \cdot \vec{\mathbf{s}}(t)\big) = \vec{\mathbf{r}}'(t) \cdot \vec{\mathbf{s}}(t) + \vec{\mathbf{r}}(t) \cdot \vec{\mathbf{s}}'(t)$
- $\frac{d}{dt}\big(\vec{\mathbf{r}}(t) \times \vec{\mathbf{s}}(t)\big) = \vec{\mathbf{r}}'(t) \times \vec{\mathbf{s}}(t) + \vec{\mathbf{r}}(t) \times \vec{\mathbf{s}}'(t)$
- $\frac{d}{dt}\big(\vec{\mathbf{r}}(f(t))\big) = f'(t)\,\vec{\mathbf{r}}'(f(t))$

Motion in space

If $\vec{\mathbf{r}}(t)$ is the vector position of a particle, then

- $\vec{\mathbf{r}}'(t) = \vec{\mathbf{v}}(t)$ is the vector velocity of the particle.
- $|\vec{\mathbf{r}}'(t)| = |\vec{\mathbf{v}}(t)| =$ speed of the particle.
- $\vec{\mathbf{r}}''(t) = \vec{\mathbf{a}}(t)$ is the vector acceleration of the particle.

We can use $\vec{\mathbf{a}}(t)$ to find the force that an object exerts: $\vec{\mathbf{F}}(t) = m\,\vec{\mathbf{a}}(t)$

Example. Suppose that a mass of 40 kg starts with init. pos'n $\langle 1, 0, 0 \rangle$, initial velocity $\langle 1, -1, 1 \rangle$ and has acceleration $\vec{\mathbf{a}}(t) = \langle 4t, 6t, 1 \rangle$.

(a) Find the position and velocity of the particle as a function of t.

(b) Determine the force that the particle exerts at time $t = 2$.

Example. Show that if a particle moves with constant speed, then the velocity and acceleration vectors are orthogonal.