Arc length

The arc length of a vector function is calculated by:

$$\int dL = \int_{t=a}^{t=b} \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2 + \frac{dz}{dt}^2}\, dt = \int_{t=a}^{t=b} |\vec{\mathbf{r}}'(t)|\, dt$$

The **arc length function** is $s(t) = \int_{u=a}^{u=t} |\vec{\mathbf{r}}'(u)|\, du$.

- We use u to parametrize the curve instead of t.
- $s(t)$ is a function of t telling the distance traveled since a.

Example. Determine the distance that a particle travels from its initial position $(1, 0, 0)$ to any point on the curve

$$\vec{\mathbf{r}}(t) = \cos t\,\vec{\mathbf{i}} + \sin t\,\vec{\mathbf{j}} + t\,\vec{\mathbf{k}}.$$

Answer: We are looking for $s(t)$ starting at $t =$ ____.

$$s(t) = \int_{u=0}^{u=t} \sqrt{\qquad\qquad\qquad\qquad}\, du =$$

The distance travelled from time 0 to time t is $s(t) =$ ______.

Reparametrization with respect to arc length

You may want to reparametrize your curve so that

$$\text{one unit in your parameter} \longleftrightarrow \text{one unit in distance}$$

To do this, we need to replace t by s.

Since we have s as a function of t, we need the inverse function!

In our example, $s = \sqrt{2}t$, so $t = \frac{s}{\sqrt{2}}$. Substituting,

$$\vec{\mathbf{r}}(s) = \cos \tfrac{s}{\sqrt{2}}\,\vec{\mathbf{i}} + \sin \tfrac{s}{\sqrt{2}}\,\vec{\mathbf{j}} + \tfrac{s}{\sqrt{2}}\,\vec{\mathbf{k}}.$$

Frenet Frame

There are many different parametrizations of any one curve.
The vectors $\vec{\mathbf{r}}(t)$, $\vec{\mathbf{v}}(t)$, $\vec{\mathbf{a}}(t)$ depend on the parameter.
But the curve itself has intrinsic properties. **At every point:**

Three natural vectors make up the Frenet frame, or TNB frame.

- $\vec{\mathbf{T}}$ The direction of the tangent vector. $\vec{\mathbf{T}}(t) = \frac{\vec{\mathbf{r}}'(t)}{|\vec{\mathbf{r}}'(t)|}$.
- $\vec{\mathbf{N}}$ The direction in which the curve is turning. $\vec{\mathbf{N}}(t) = \frac{\vec{\mathbf{T}}'(t)}{|\vec{\mathbf{T}}'(t)|}$.
- $\vec{\mathbf{B}}$ The third vector that completes $\perp$ basis. $\vec{\mathbf{B}}(t) = \vec{\mathbf{T}}(t) \times \vec{\mathbf{N}}(t)$

A number that tells how bendy or twisty the curve is.

Definition: The **curvature** $\kappa(t)$ of a curve ("kappa") tells how quickly $\vec{\mathbf{T}}$ is changing *with respect to distance traveled.*

$$\kappa = \left|\frac{d\vec{\mathbf{T}}}{ds}\right| \overset{\text{chain rule}}{=} \left|\frac{\frac{d\vec{\mathbf{T}}}{dt}}{\frac{ds}{dt}}\right| = \frac{|\vec{\mathbf{T}}'(t)|}{|\vec{\mathbf{r}}'(t)|} \overset{\text{algebra}}{=} \frac{|\vec{\mathbf{r}}'(t) \times \vec{\mathbf{r}}''(t)|}{|\vec{\mathbf{r}}'(t)|^3}.$$

The circle that lies along the curve has radius $1/\kappa$. (**!**)

Curvature

Example. Determine the vectors of the Frenet frame and the curvature of the curve $\vec{\mathbf{r}}(t) = \langle \cos t, \sin t, t \rangle$.

Frenet frame: We need $\vec{\mathbf{r}}'(t) = \langle -\sin t, \cos t, 1 \rangle$ and $|\vec{\mathbf{r}}'(t)| =$ ______.
Then $\vec{\mathbf{T}}(t) = \frac{\vec{\mathbf{r}}'(t)}{|\vec{\mathbf{r}}'(t)|} = \langle \frac{-\sin t}{\sqrt{2}}, \frac{\cos t}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$,
From this we find that $\vec{\mathbf{T}}'(t) = \langle \frac{-\cos t}{\sqrt{2}}, \frac{-\sin t}{\sqrt{2}}, 0 \rangle$, so $|\vec{\mathbf{T}}'(t)| =$ ______.
Then $\vec{\mathbf{N}}(t) = \frac{\vec{\mathbf{T}}'(t)}{|\vec{\mathbf{T}}'(t)|} = \langle -\cos t, -\sin t, 0 \rangle$.

Now $\vec{\mathbf{B}}(t) = \vec{\mathbf{T}}(t) \times \vec{\mathbf{N}}(t) = \frac{1}{\sqrt{2}} \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ -\sin t & \cos t & 1 \\ -\cos t & -\sin t & 0 \end{vmatrix} =$

The curvature $\kappa(t) = \frac{|\vec{\mathbf{T}}'(t)|}{|\vec{\mathbf{r}}'(t)|} =$

Question: Should $\kappa(t)$ be a constant?

Components of Acceleration

The curvature tells us about the centripetal force we feel.

Key idea: Understand $\vec{\mathbf{a}}$ in terms of the Frenet frame:
How much of the acceleration is toward $\vec{\mathbf{T}}$, $\vec{\mathbf{N}}$, and $\vec{\mathbf{B}}$?

Differentiate $\vec{\mathbf{v}}(t) = v(t)\vec{\mathbf{T}}(t)$. (magnitude (speed) times unit direction)

$$\vec{\mathbf{a}} = v'\vec{\mathbf{T}} + v\vec{\mathbf{T}}' = v'\vec{\mathbf{T}} + v(|\vec{\mathbf{T}}'|\vec{\mathbf{N}}) = v'\vec{\mathbf{T}} + \kappa v^2\vec{\mathbf{N}}.$$

- All acceleration is toward $\vec{\mathbf{T}}$ and $\vec{\mathbf{N}}$. (Not to $\vec{\mathbf{B}}$.)
- a_T Toward $\vec{\mathbf{T}}$: $a_T = v'$ is rate of change of speed.
- a_N Toward $\vec{\mathbf{N}}$: $a_N = \kappa v^2$. Curvature times speed squared!

Solve for a_T, a_N in terms of $\vec{\mathbf{r}}(t)$.

First, $\vec{\mathbf{v}} \cdot \vec{\mathbf{a}} = v\vec{\mathbf{T}} \cdot (v'\vec{\mathbf{T}} + \kappa v^2\vec{\mathbf{N}}) = vv'\vec{\mathbf{T}} \cdot \vec{\mathbf{T}} + \kappa v^3\vec{\mathbf{T}} \cdot \vec{\mathbf{N}} = vv'$

So $a_T = v' = \frac{\vec{\mathbf{v}}\cdot\vec{\mathbf{a}}}{v} = \frac{\vec{\mathbf{r}}'(t)\cdot\vec{\mathbf{r}}''(t)}{|\vec{\mathbf{r}}'(t)|}$ and $a_N = \kappa v^2 = \frac{|\vec{\mathbf{r}}'(t)\times\vec{\mathbf{r}}''(t)|}{|\vec{\mathbf{r}}'(t)|}$ Nice symmetry!

Example. Find tang'l, normal comp's of acceleration for $\vec{\mathbf{r}} = \langle t, 2t, t^2\rangle$.