Limits

Function of one variable

$$\lim_{x \to a} f(x) = L$$

Visually:

Interpretation:

However you approach $x = a$, the value $f(x)$ **always** approaches L.

Mathematically:

No matter how close to $y = L$ you insist you must be (ε-close),

There is a way to choose a range δ around $x = a$ to ensure that

All values within δ of a give function values within ε of L.

Function of several variables

$$\lim_{(x,y) \to (a,b)} f(x, y) = L$$

Visually:

Interpretation:

However you approach $(x, y) = (a, b)$, the value $f(x, y)$ **always** approaches L.

Mathematically:

No matter how close to $z = L$ you insist you must be (ε-close),

There is a way to choose a radius δ around $(x, y) = (a, b)$ to ensure that

All values within δ of (a, b) give function values within ε of L.

How might we convince ourselves that a limit exists?

Question: Why not take 1D limits along lines headed toward (a, b)?

Answer: Because looks can be deceiving!

Key idea: When limits along different paths do not agree, limit DNE.

Example. Show that $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ does not exist.

Along the x-axis:

Along the y-axis:

The limits along different paths do not agree, so the limit DNE.

More lines of thought

Example. Does the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ exist?

Along the x-axis: $\displaystyle\lim_{(x,0)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{x\cdot 0}{x^2+0} =$

Along the y-axis: $\displaystyle\lim_{(0,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{y\to 0} \frac{0\cdot y}{0+y^2} =$

Along the line $y = x$:

Answer:

Example. Does the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2}$ exist?

Along the x-axis: $\displaystyle\lim_{(x,0)\to(0,0)} \frac{xy^2}{x^2+y^2} = \lim_{x\to 0} \frac{x\cdot 0}{x^2+0} =$

Along the y-axis: $\displaystyle\lim_{(0,y)\to(0,0)} \frac{xy^2}{x^2+y^2} = \lim_{y\to 0} \frac{0\cdot y^2}{0+y^2} =$

Along any line $y = mx$: $\displaystyle\lim_{(x,mx)\to(0,0)} \frac{xy^2}{x^2+y^2} =$

Answer:

When **DO** we know a limit exists?

A function $f(x, y)$ is **continuous** at (a, b) if $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$.

- The function exists at (a, b).
- The limit exists at (a, b).
- The two values are equal.

Continuity is a given in certain cases:

- A polynomial is continuous everywhere.
- A rational function is continuous on its domain.
- The composition of two continuous functions is continuous.

Example. $\arctan(y/x)$ is continuous on its domain since $\arctan(t)$ is continuous and y/x is a rational function of x and y.

Consequence: If we know $f(x, y)$ is continuous at (a, b), then $\lim_{(x,y)\to(a,b)} f(x, y)$ exists!

Partial derivatives

Suppose f is a function of both x and y.

- Fix $y = b$ and let only x vary.
- Then $f(x, b)$ is a function of one variable.
- We can take its derivative with respect to x.

This is the **partial derivative of f with respect to x**. We write:

$$f_x(x,y) \quad \textbf{or} \quad \frac{\partial f}{\partial x} \quad \textbf{or} \quad \frac{\partial}{\partial x} f(x,y) \quad \textbf{or} \quad \frac{\partial z}{\partial x} \quad \textbf{or} \quad D_x f.$$

★ **Idea:** Treat other variables as constants, differentiate normally. ★

Example. Let $f(x, y) = x^3 + x^2y^3 - 2y^2$. Find $f_x(2, 1)$ and $f_y(2, 1)$.

More examples

Example. Let $g(x, y) = \sin \frac{x}{1+y}$. Find $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial y}$.

Example. If $x^3 + y^3 + z^3 + 6xyz = 1$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Answer: Here z is defined implicitly as a function of x and y.

$\frac{\partial}{\partial x}\left(x^3 + y^3 + z^3 + 6xyz\right) = \frac{\partial}{\partial x}(0)$

$$\frac{\partial z}{\partial x} = \frac{-(3x^2 + 6yz)}{3z^2 + 6xy} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{-(3y^2 + 6xz)}{3z^2 + 6xy}$$