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*Mathematically:*

No matter how close to  $y = L$  you insist you must be ( $\varepsilon$ -close),  
There is a way to choose a range  $\delta$  around  $x = a$  to ensure that

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**Example.** Show that  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{x^2 + y^2}$  does not exist.

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**Consequence:** If we know  $f(x, y)$  is continuous at  $(a, b)$ , then  $\lim_{(x,y) \rightarrow (a,b)} f(x, y)$  exists!

## Partial derivatives

Suppose  $f$  is a function of both  $x$  and  $y$ .

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This is the **partial derivative of  $f$  with respect to  $x$** . We write:

$$f_x(x, y) \quad \text{or} \quad \frac{\partial f}{\partial x} \quad \text{or} \quad \frac{\partial}{\partial x} f(x, y) \quad \text{or} \quad \frac{\partial z}{\partial x} \quad \text{or} \quad D_x f.$$

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★ **Idea:** Treat other variables as constants, differentiate normally. ★

**Example.** Let  $f(x, y) = x^3 + x^2y^3 - 2y^2$ . Find  $f_x(2, 1)$  and  $f_y(2, 1)$ .

## More examples

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$$\frac{\partial z}{\partial x} = \frac{-(3x^2 + 6yz)}{3z^2 + 6xy} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{-(3y^2 + 6xz)}{3z^2 + 6xy}$$