Definition of the directional derivative

Partial derivatives allow us to see how fast a function changes.

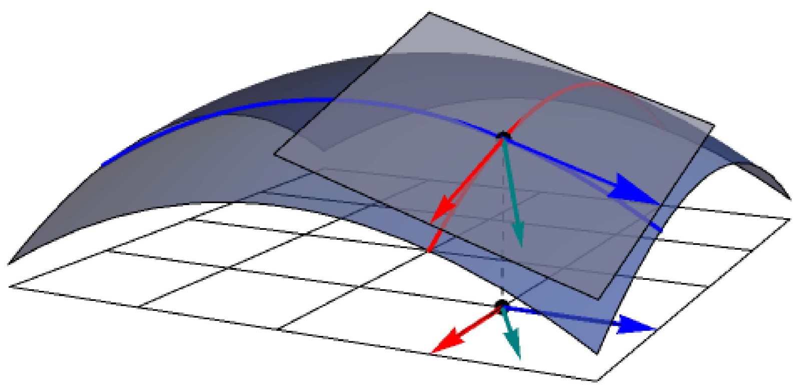
$D_x f = f_x(x, y)$ is the rate of change of f in the x-direction. Toward $\vec{\mathbf{i}} = (1, 0)$

$D_y f = f_y(x, y)$ is the rate of change of f in the y-direction. Toward $\vec{\mathbf{j}} = (0, 1)$

Question: How fast is $f(x, y)$ changing in **some other direction**?

What does that even mean?

Question: What is the rate of change of f toward *unit vector* $\vec{\mathbf{u}} = (a, b) = (\cos\theta, \sin\theta)$?



Definition: The directional derivative of f in the direction of $\vec{\mathbf{u}}$ is

$$D_{\vec{\mathbf{u}}} f(x, y) = f_x(x, y)\, a + f_y(x, y)\, b.$$

Directional derivative example

Example. Find $D_{\vec{\mathbf{u}}}f$ if $f(x, y) = x^3 - 3xy + 4y^2$ and $\vec{\mathbf{u}}$ is the unit vector in the xy-plane at angle $\theta = \pi/6$.

Answer: First, find the vector $\vec{\mathbf{u}} =$

Next, find the partial derivatives:

$\frac{\partial f}{\partial x} =$ $\frac{\partial f}{\partial y} =$.

We conclude that $D_{\vec{\mathbf{u}}}f(x, y) =$

Example. Calculate $D_{\vec{\mathbf{u}}}f(1, 2)$ and interpret this answer.

$$D_{\vec{\mathbf{u}}}f(1,2) = (3 \cdot 1 - 3 \cdot 2)\tfrac{\sqrt{3}}{2} + (-3 \cdot 1 + 8 \cdot 2)\tfrac{1}{2}$$
$$= \frac{13 - 2\sqrt{3}}{2} \approx 3.9$$

Interpretation: One unit step in the $\vec{\mathbf{u}}$ direction increases $f(x, y)$ by approximately 3.9 units.

Motivating the gradient

Notice that $D_{\vec{\mathbf{u}}}f = f_x a + f_y b$.
Rewrite as: $D_{\vec{\mathbf{u}}}f = \langle f_x, f_y \rangle \cdot \langle a, b \rangle$.

Definition: The vector $\langle f_x, f_y \rangle = f_x \vec{\mathbf{i}} + f_y \vec{\mathbf{j}}$ is called the **gradient** of f.
We write ∇f or grad f.

So an alternate way to write $D_{\vec{\mathbf{u}}}f(x, y)$ is $\nabla f(x, y) \cdot \vec{\mathbf{u}}$.

The gradient is also defined for functions of more than two variables.
For example, for a function of three variables, $f(x, y, z)$,

$$\nabla f = \langle f_x, f_y, f_z \rangle = f_x \vec{\mathbf{i}} + f_y \vec{\mathbf{j}} + f_z \vec{\mathbf{k}}$$

$$\text{and } D_{\vec{\mathbf{u}}}f = \nabla f \cdot \vec{\mathbf{u}}$$

Applying ∇f

Example. Let $f(x, y, z) = x \sin(yz)$. Find the directional derivative of f at $(1, 3, 0)$ in the direction $\vec{\mathbf{v}} = \vec{\mathbf{i}} + 2\vec{\mathbf{j}} - \vec{\mathbf{k}}$.

Step back. What do we want to calculate?

Game Plan:

- Find a unit vector in the direction of $\vec{\mathbf{v}}$.
- Find ∇f, plug in $(1, 3, 0)$.
- Take the dot product.

Therefore $D_{\vec{\mathbf{u}}} f(1, 3, 0) =$

Interpretation?

An important interpretation of the gradient

Question: Given a function $f(x, y)$ and a point (x_0, y_0),
(or a function $f(x, y, z)$ and a point (x_0, y_0, z_0)),
in which direction is the function increasing the *fastest*?
And how fast is the function increasing in that direction?

Answer: At a rate of $|\nabla f(x_0, y_0)|$, in the direction of $\nabla f(x_0, y_0)$!!

But why?!?

$$\begin{aligned} D_{\vec{\mathbf{u}}} f &= \nabla f \cdot \vec{\mathbf{u}} = |\nabla f|\,|\vec{\mathbf{u}}| \cos(\theta) \\ &= |\nabla f| \cos(\theta) \end{aligned}$$

Question: For what angle θ is this maximized? And what is the max?

Answer:

Consequence: ∇f represents the direction of fastest increase of f.

Visualization of the gradient

∇f represents the direction of fastest increase of f.

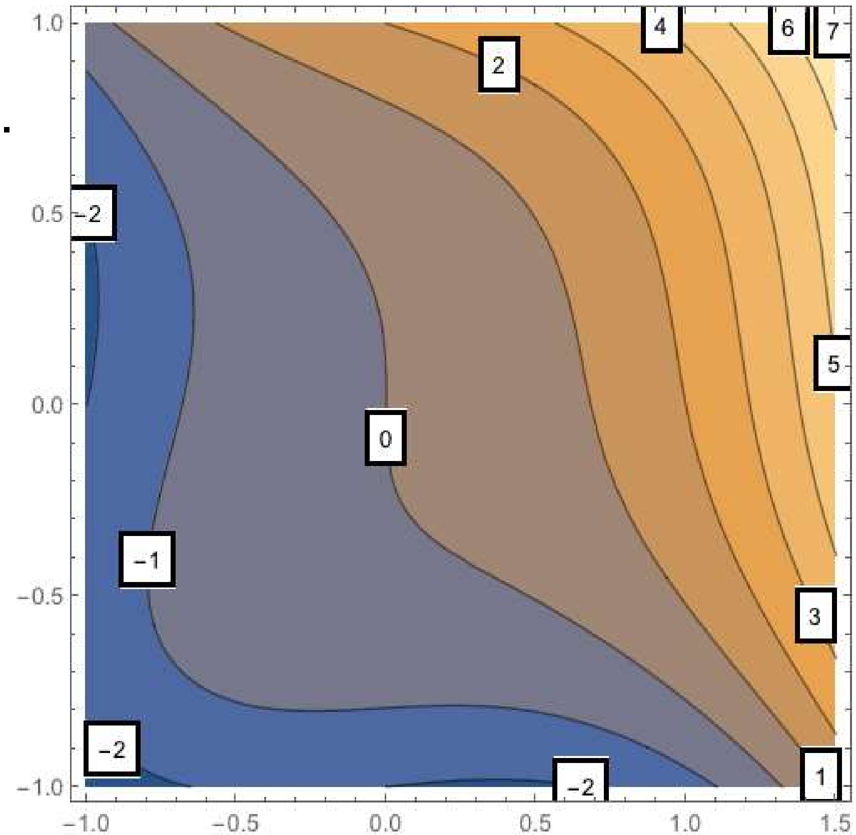
We can understand this graphically through the contour map.

- At (x_0, y_0), the vector $\nabla f(x_0, y_0)$ is perpendicular to the level curves of f.

Why?

- Along a level curve, f is constant.
- The fastest change should be perpendicular to the level curve.

♡ Connecting along this path gives ♡
♡ the **path of steepest ascent**. ♡
Chloe says "hi".



Tangent planes to level surfaces

Functions of two variables	**Functions of three variables**
A *level curve* $f(x,y)=c$	A *level surface* $F(x,y,z)=c$
$\nabla f \longleftrightarrow$ fastest increase	$\nabla F \longleftrightarrow$ fastest increase
So: ∇f is $\perp$ (to tangent line) to level curve at (x_0, y_0)	so ∇F is $\perp$ (to tangent plane) to level surface at (x_0, y_0, z_0)

$\nabla F(x_0, y_0, z_0)$ is the **normal vector** to the level surface at (x_0, y_0, z_0).

This means: The equation of THE tangent plane to THE level surface passing through the point (x_0, y_0, z_0) is

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

Also: For any curve $\vec{\mathbf{r}}(t) = (x(t), y(t), z(t))$ on the level surface,

$$F\big(x(t), y(t), z(t)\big) = k \quad \overset{\text{chain}}{\Longrightarrow} \quad \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = 0,$$

which means $\nabla F \perp \vec{\mathbf{r}}'(t) = 0$.