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Partial derivatives allow us to see how fast a function changes.

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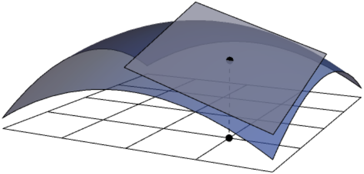

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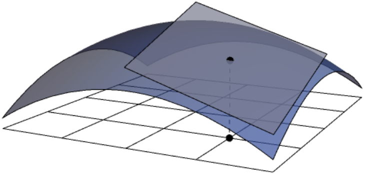

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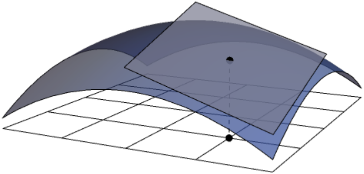

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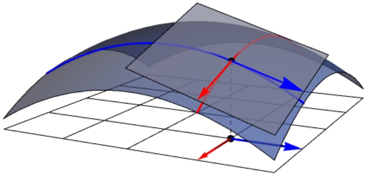

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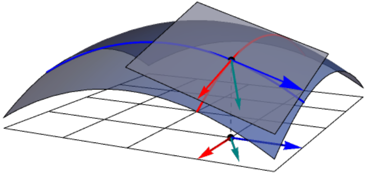

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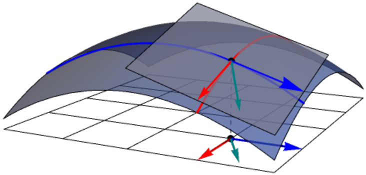

Definition: The directional derivative of f in the direction of $\vec{\mathbf{u}}$ is

$$D_{\vec{\mathbf{u}}} f(x, y) = f_x(x, y)\, a + f_y(x, y)\, b.$$

Directional derivative example

Example. Find $D_{\vec{\mathbf{u}}}f$ if $f(x, y) = x^3 - 3xy + 4y^2$ and $\vec{\mathbf{u}}$ is the unit vector in the xy-plane at angle $\theta = \pi/6$.

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$$D_{\vec{\mathbf{u}}}f(1,2) = (3 \cdot 1 - 3 \cdot 2)\tfrac{\sqrt{3}}{2} + (-3 \cdot 1 + 8 \cdot 2)\tfrac{1}{2}$$
$$= \frac{13 - 2\sqrt{3}}{2} \approx 3.9$$

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Interpretation: One unit step in the $\vec{\mathbf{u}}$ direction increases $f(x,y)$ by approximately 3.9 units.

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The gradient is also defined for functions of more than two variables.
For example, for a function of three variables, $f(x, y, z)$,

$$\nabla f = \langle f_x, f_y, f_z \rangle = f_x \vec{\mathbf{i}} + f_y \vec{\mathbf{j}} + f_z \vec{\mathbf{k}}$$

$$\text{and } D_{\vec{\mathbf{u}}} f = \nabla f \cdot \vec{\mathbf{u}}$$

Applying ∇f

Example. Let $f(x, y, z) = x\sin(yz)$. Find the directional derivative of f at $(1, 3, 0)$ in the direction $\vec{\mathbf{v}} = \vec{\mathbf{i}} + 2\vec{\mathbf{j}} - \vec{\mathbf{k}}$.

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Interpretation?

stop

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Question: Given a function $f(x, y)$ and a point (x_0, y_0),

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Consequence: ∇f represents the direction of fastest increase of f.

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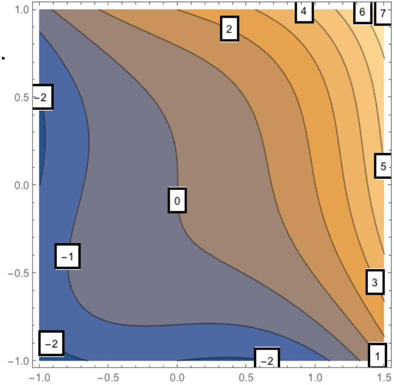

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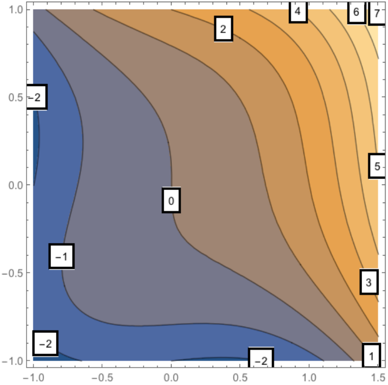

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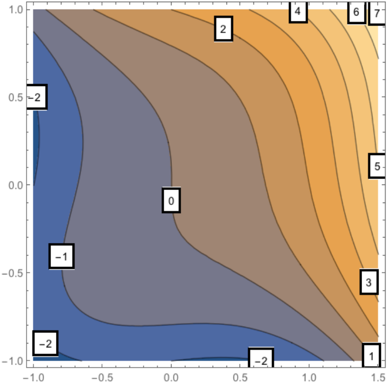

Why?

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♡ Connecting along this path gives ♡
♡ the **path of steepest ascent**. ♡
Chloe says "hi".

Tangent planes to level surfaces

Functions of two variables

A *level curve* $f(x, y) = c$

Functions of three variables

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which means $\nabla F \perp \vec{\mathbf{r}}'(t) = 0$.