Optimization subject to constraints

The method of Lagrange multipliers is an alternative way to find maxima and minima of a function $f(x, y, z)$ subject to a given constraint $g(x, y, z) = k$.

Motivating Example. Suppose you are trying to find the maximum and minimum value of $f(x, y) = y - x$ when we only consider points on the curve $g(x, y) = x^2 + 4y^2 = 36$.

What should we do?

Introducing Lagrange multipliers

For functions of two variables:

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The method of Lagrange multipliers

To find the maxima and minima of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ (as long as $\nabla g \neq 0$ on this constraint)

- Solve for all tuples (x, y, z, λ) such that

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  - *In words:* the gradient of f is parallel to the gradient of g.

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- Careful about when this applies. $(\nabla g \neq 0)$

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- Multiply Eq. (2) by y, Eq. (3) by z, simplify.

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Solution?