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Then define $\displaystyle\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA.$

(Which we know exists)

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Some regions work either way. Choose based on $f(x,y)$.

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Steps:

1. Plot the curves (Draw a picture!)
2. Find points of intersection
3. Determine order of integration
4. Determine "upper" and "lower" functions, other bounds
5. Do the integrals.

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We calculate $\displaystyle\int_{y=-2}^{y=4} \int_{x=\frac{y^2-6}{2}}^{x=y+1} xy\,dx\,dy =$

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$$= \tfrac{1}{2}\left[-\tfrac{y^6}{24} + y^4 + \tfrac{2}{3}y^3 - 4y^2\right]_{y=-2}^{y=4} = 36$$

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Sometimes you need to find D and f from the problem statement.

Example. Set up the integral that finds the volume of the solid bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.

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Our function is $f(x, y) = z = 2 - x - 2y$, and our integral is

$$\int_{x=0}^{x=1} \int_{y=x/2}^{y=1-x/2} (2 - x - 2y)\, dy\, dx$$

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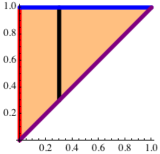

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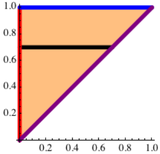

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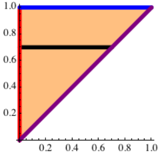

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Property. Suppose that $D = D_1 \cup D_2$ (where D_1 and D_2 don't overlap). Then

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Consequence: This gives a crude approximation for the integral.

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Suppose you have a 2D sheet of metal (a **lamina**) where density varies over the sheet.

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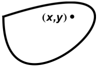
mass density function $\rho(x, y)$ (mass per unit area) 	

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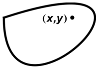
mass density function $\rho(x, y)$ (mass per unit area) 	The total mass of the object is $$m = \iint_D \rho(x, y)\, dA$$

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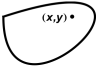
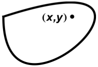
mass density function $\rho(x, y)$ (mass per unit area) 	The total mass of the object is $$m = \iint_D \rho(x, y)\, dA$$
charge density function $\sigma(x, y)$ (charge per unit area) 	

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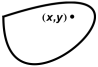
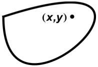
mass density function $\rho(x, y)$ (mass per unit area) 	The total mass of the object is $$m = \iint_D \rho(x, y)\, dA$$
charge density function $\sigma(x, y)$ (charge per unit area) 	The total charge on the object is $$Q = \iint_D \sigma(x, y)\, dA$$

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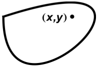
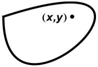
mass density function $\rho(x,y)$ (mass per unit area) 	The total mass of the object is $$m = \iint_D \rho(x,y)\,dA$$
charge density function $\sigma(x,y)$ (charge per unit area) 	The total charge on the object is $$Q = \iint_D \sigma(x,y)\,dA$$

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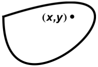
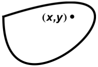
mass density function $\rho(x, y)$ (mass per unit area) 	The total mass of the object is $$m = \iint_D \rho(x, y)\, dA$$
charge density function $\sigma(x, y)$ (charge per unit area) 	The total charge on the object is $$Q = \iint_D \sigma(x, y)\, dA$$

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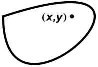
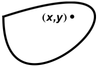
mass density function $\rho(x, y)$ (mass per unit area) 	The total mass of the object is $$m = \iint_D \rho(x, y)\, dA$$
charge density function $\sigma(x, y)$ (charge per unit area) 	The total charge on the object is $$Q = \iint_D \sigma(x, y)\, dA$$

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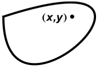
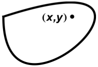
mass density function $\rho(x, y)$ (mass per unit area) 	The total mass of the object is $$m = \iint_D \rho(x, y)\, dA$$
charge density function $\sigma(x, y)$ (charge per unit area) 	The total charge on the object is $$Q = \iint_D \sigma(x, y)\, dA$$

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$$= \int_{x=0}^{x=1} (6x + 4x^2)\, dx = 3x^2 + \frac{4}{3}x^3\Big]_{x=0}^{x=1} = \frac{13}{3}$$