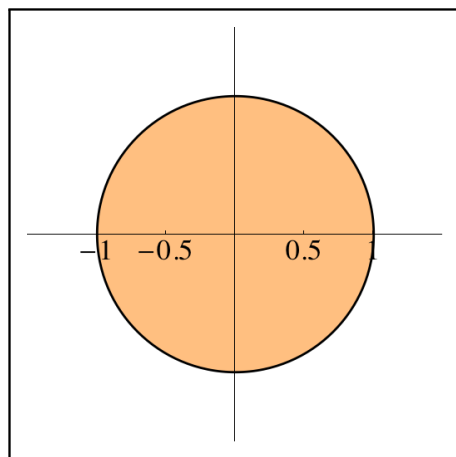
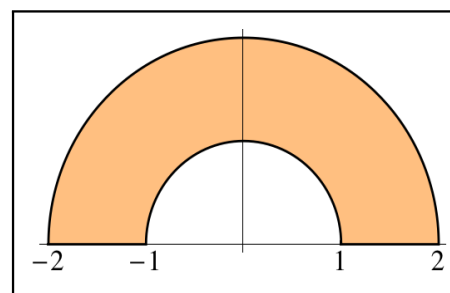


Some regions *demand* polar coordinates

Some regions are best described in polar coordinates:



Parts of circles



Parts of annuli

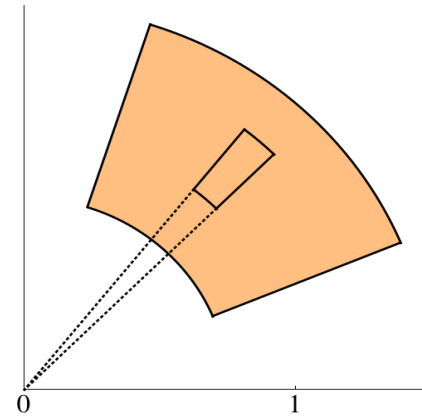
These are called **polar rectangles** because they have the form

$$\left\{ \begin{array}{l} a \leq r \leq b \\ \alpha \leq \theta \leq \beta \end{array} \right\} \text{ for constants } \left\{ \begin{array}{l} a, b \\ \alpha, \beta \end{array} \right\}$$

Polar coordinates

Goal: Convert a double integral involving x 's and y 's into a double integral involving r 's and θ 's.

Important: Using “polar slices” introduces a complication.



In this picture, dA is **not** $dr d\theta$.

The radial component is _____ and the circular component is _____.

This means $dA =$ _____. (How to remember?)

Consequence: To calculate $\iint_D f(x, y) dA$, when D is best described in polar coordinates, calculate

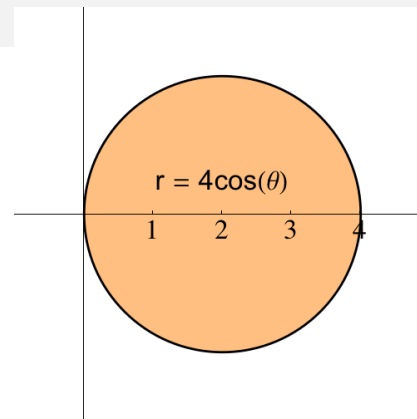
$$\iint_D f(r \cos \theta, r \sin \theta) r dr d\theta$$

Polar coordinate example

Example. Find the area inside
 $r = 4 \cos \theta$ from $\theta = \frac{\pi}{4}$ to $\frac{\pi}{2}$.

This region is defined as $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$
 and _____ $\leq r \leq$ _____.

We use $A = \iint dA$:



★ Along the way, we had $\int_{\alpha}^{\beta} \frac{r^2}{2} d\theta \dots\dots\dots$

Application: Density (p. 692/723)

Example. The density of a point on a semicircular lamina of radius a is proportional to the distance from the center of the circle. Find the mass of the lamina.

Solution. Draw a picture!

Setup: Let the lamina be the upper half of the circle $x^2 + y^2 = a^2$, which is a polar rectangle:

The density function can be written:

$$\rho(x, y) = K\sqrt{x^2 + y^2}$$

The total mass is $m = \iint_D \rho(x, y) dA$

$$\dots = \int_{\theta=0}^{\theta=\pi} K \frac{a^3}{3} d\theta = \left[K \frac{a^3}{3} \theta \right]_{\theta=0}^{\theta=\pi} = \frac{\pi K a^3}{3}.$$

Changing from (x, y) to (r, θ)

Example. Calculate $\int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (x^3 + xy^2) dy dx$ (hint for polar!)

Solution. Draw a picture!

Notice that this is the polar rectangle

$$0 \leq r \leq 3 \text{ and } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}.$$

Rewrite the integral as

$$\begin{aligned} & \iint_D (r^3 \cos^3 \theta + r^3 \cos \theta \sin^2 \theta) r dr d\theta \\ &= \iint_D r^4 \cos \theta (\cos^2 \theta + \sin^2 \theta) dr d\theta = \iint_D r^4 \cos \theta dr d\theta \\ &= \left(\int_{r=0}^{r=3} r^4 dr \right) \cdot \left(\int_{\theta=-\pi/2}^{\theta=\pi/2} \cos \theta d\theta \right) \\ &= \left[\frac{r^5}{5} \right]_{r=0}^{r=3} \cdot [\sin \theta]_{\theta=-\pi/2}^{\theta=\pi/2} = \left(\frac{3^5}{5} - 0 \right) \cdot (1 - (-1)) = \frac{2 \cdot 3^5}{5} \end{aligned}$$

Volume example

Example. Find the volume of the solid under the paraboloid $z = x^2 + y^2$, above the xy -plane, and inside the cylinder $(x - 1)^2 + y^2 = 1$.

Plan of attack: Draw a picture!

- ▶ Notice circley thingees. Think: Possible Polar Problem.
- ▶ Convert the given information to polar equations using

$$(x, y) = (r \cos \theta, r \sin \theta):$$

$$(x-1)^2 + y^2 = 1 \rightsquigarrow (r \cos \theta - 1)^2 + (r \sin \theta)^2 = 1 \rightsquigarrow r = 2 \cos \theta.$$

$$(x^2 + y^2) \rightsquigarrow r^2.$$

- ▶ Set up the polar integral.

$$\text{So } \iint_D (x^2 + y^2) dA = \iint_D r^2 r dr d\theta.$$

- ▶ Integrate.....