Generalizations of Polar Coordinates

When 2-dim'l regions D have radial flavors, we use polar coordinates. When 3-dim'l regions E have radial flavors, there are two choices:

Cylindrical coordinates

A point can have coords (r, θ, z) :

Spherical coordinates

A point can have coords (ρ, θ, ϕ) .

 (r, θ) are polar coords of xy-plane (r is the distance from the z-axis) and z is the "distance" to xy-plane

Useful when problems involve symmetry about an axis. Cylinder, Paraboloids, Cones w/flat bases ρ is the distance from the origin

 θ is same as in polar

 ϕ is \angle between +z axis and \overline{OP} .

Useful when problems involve symmetry about a point Spheres, Cones with curved bases.

Converting from cartesian coordinates

Cylindrical coordinates

Conversion:

$$x = r \cos \theta$$
 $y = r \sin \theta$ $z = z$
 $r^2 = x^2 + y^2$ $\tan \theta = \frac{y}{x}$ $z = z$

$$dV = r dr d\theta dz$$

Spherical coordinates

Conversion:
$$z = r \cos \theta$$
 $z = r \sin \theta$ $z = z$ $z = \rho \cos \phi$ $z = \rho \sin \phi \cos \theta$ $z = z$ $z = \rho \cos \phi$ $z = \rho \sin \phi \sin \phi$ $z = z$ $z = \rho \cos \phi$ $z = \rho \sin \phi \sin \phi$ $z = \rho \sin \phi \cos \phi$ $z = \rho \sin \phi \sin \phi$

$$dV = \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi$$

Practice

Cylindrical coordinates

Practice changing coordinates:

$$(r, \theta, z) = (2, \frac{2\pi}{3}, 1); (x, y, z) = (3, -3, 7)$$

Identify cyl. coord. equations:

2.
$$r = 2$$
; $z = r^2$; $r^2 - 2z^2 = 4$

3. Sketch
$$r^2 \le z \le 2 - r^2$$

Convert to cylindrical coordinates

4.
$$x^2 + y^2 + z^2 = 2$$
; $x^2 + y^2 = 2y$

5. Give solid between $x^2 + y^2 = 1$ and $x^2 + y^2 + z^2 = 4$.

6.
$$\begin{cases} -2 \le x \le 2 \\ -\sqrt{4 - x^2} \le y \le \sqrt{4 - x^2} \\ \sqrt{x^2 + y^2} \le z \le 2 \end{cases}$$

Spherical coordinates

Practice changing coordinates:

$$(\rho, \theta, \phi) = (2, \frac{\pi}{4}, \frac{\pi}{3}); (x, y, z) = (-1, 1, \sqrt{6})$$

Identify sph. coord. equations:

2.
$$\phi = \frac{\pi}{3}$$
; $\rho \sin \phi = 2$; $\rho = 2 \cos \phi$

3. Sketch
$$(2 \le \rho \le 3 \& \frac{\pi}{2} \le \phi \le \pi)$$
 Sketch $(0 \le \phi \le \frac{\pi}{3} \& \rho \le 2)$

Convert to spherical coordinates

4.
$$z = x^2 + y^2$$
; $z = x^2 - y^2$

5. Give solid inside $x^2 + y^2 + z^2 = 4$, above xy-plane, below $z = \sqrt{x^2 + y^2}$.

6.
$$\begin{cases} 0 \le x \le 1 \\ 0 \le y \le \sqrt{1 - x^2} \\ \sqrt{x^2 + y^2} \le z \le \sqrt{2 - x^2 - y^2} \end{cases}$$