Area under a parametric curve

Given y = f(x), the area under the curve from x = a to x = b is Area $= \int_{x=a}^{x=b} \leftarrow \text{right endpoint} = \int_{t=\alpha}^{t=\beta} \leftarrow \text{right endpoint} = \int_{t=\alpha}^{t=\beta} \frac{g(t)f'(t)dt}{g(t)f'(t)dt}$

Example. Find the area under one arch of the cycloid $\begin{cases} x = r(\theta - \sin \theta) \\ y = r(1 - \cos \theta) \end{cases}$ (Here, *r* is a constant and θ is the parameter.)

Plot it to see the shape. One arch has range $___ \leq \theta \leq ___$.

$$A = \int y \, dx = \int r(1 - \cos \theta) r(1 - \cos \theta) \, d\theta$$
$$= r^2 \int (1 - 2\cos \theta + \cos^2 \theta) \, d\theta$$

Area inside a polar curve

For cartesian functions y = f(x), calculate area as $A = \int dA = \int y \, dx$.

What about the area "*inside a curve*" best described as a polar function $r = f(\theta)$?

• We still use
$$A = \int dA$$
.

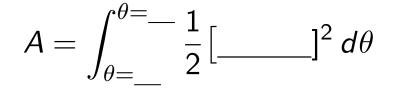
But the formula for dA is different.

$$A = \int_{\theta=a}^{\theta=b} \frac{1}{2} \left[f(\theta) \right]^2 d\theta = \int_a^b \frac{1}{2} r^2 d\theta$$

Polar land How much area is swept out by a little slice?

Example. What is the area inside one loop of the four-leaved rose $r = \cos 2\theta$?

- ▶ What are the bounds on θ ?
- For which θ does the curve pass through the origin?



Inside yet Outside

Example. Calculate the area inside the curve $r = 3 \sin \theta$ and outside the curve $r = \sin \theta + 1$.

First: Draw a picture!

- What are these curves?
- ► Where do they intersect?

Now calculate: $\int_{\alpha_{-}}^{\theta=} \left[\left(\frac{1}{2} (3\sin\theta)^2 \right) - \left(\frac{1}{2} (\sin\theta + 1)^2 \right) \right] d\theta$ $=\frac{1}{2}\int_{\theta=}^{\theta=}$ $\left[9\sin^2\theta - (\sin^2\theta + 2\sin\theta + 1)\right]d\theta$ $= \frac{1}{2} \int_{\theta}^{\theta} \frac{1}{2} \left[8 \sin^2 \theta - 2 \sin \theta - 1 \right] d\theta$ $=\frac{1}{2}\int_{\theta}^{\theta} \left[4-4\cos 2\theta-2\sin \theta-1\right]d\theta$ $=\frac{1}{2}\int_{\theta=}^{\theta=} [3-4\cos 2\theta - 2\sin \theta] d\theta$ $=\left[\frac{3\theta}{2}-\sin 2\theta+\cos \theta\right]_{\pi/6}^{5\pi/6}$ $=\frac{3}{2}\left(\frac{5\pi}{6}-\frac{\pi}{6}\right)-\left(\sin\frac{10\pi}{6}-\sin\frac{2\pi}{6}\right)+\left(\cos\frac{5\pi}{6}-\cos\frac{\pi}{6}\right)=\cdots$ $=\pi + \sqrt{3} - \sqrt{3} = \pi$

Arc length of a parametric curve

To find the arc length of a parametric curve, think $L = \int dL$. How much arc length dL does the curve traverse in one time unit dt?

Draw a right triangle:

To see $dL = \sqrt{dx^2 + dy^2}$.

How are each related to dt?

$$dL = \sqrt{\left[\frac{dx}{dt} \cdot dt\right]^2 + \left[\frac{dy}{dt} \cdot dt\right]^2}$$
$$L = \int_{t=\alpha}^{t=\beta} \sqrt{\left[f'(t)\right]^2 + \left[g'(t)\right]^2} dt$$
$$\int x = \sin 2t$$

Example. Find the arc length for the parametric curve $\int y = \cos 2t$ for $0 \le t \le 2\pi$. What do we expect? What is this curve? $\int_{t=0}^{t=2\pi} \sqrt{[f'(t)]^2 + [g'(t)]^2} dt =$

Arc length of a polar curve

For a polar curve, calculate dL in terms of dr and $d\theta$:

The right triangle shows:

$$dL = \sqrt{(r \, d\theta)^2 + (dr)^2}$$

Take θ as the parameter to get

$$dL = \sqrt{\left[r \, d\theta\right]^2 + \left[\frac{dr}{d\theta} \, d\theta\right]^2} \text{ so } L = \int_{\theta=\alpha}^{\theta=\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$$

Alternatively, use the parametrization
$$\begin{cases} x = f(\theta) \cos \theta \\ y = f(\theta) \sin \theta \end{cases}$$

$$\blacktriangleright \text{ Then } L = \int_{\theta=\alpha}^{\theta=\beta} \sqrt{\left[\frac{d}{d\theta}(f(\theta)\cos\theta)\right]^2 + \left[\frac{d}{d\theta}(f(\theta)\sin\theta)\right]^2} \, d\theta$$

$$\ldots \text{ take derivatives \& do the algebra } \ldots$$

$$L = \int_{\theta=\alpha}^{\theta=\beta} \sqrt{\left[f(\theta)\right]^2 + \left[\frac{d}{d\theta}(f(\theta))\right]^2} \, d\theta$$