Area under a parametric curve

Given $y = f(x)$, the area under the curve from $x = a$ to $x = b$ is

$$\text{Area} = \int_{x=a \;\leftarrow\; \text{left endpoint}}^{x=b \;\leftarrow\; \text{right endpoint}} y\,dx \quad = \quad \int_{t=\alpha \;\leftarrow\; \text{left endpoint}}^{t=\beta \;\leftarrow\; \text{right endpoint}} g(t)\,f'(t)\,dt$$

Example. Find the area under one arch of the cycloid $\begin{cases} x = r(\theta - \sin\theta) \\ y = r(1 - cos\theta) \end{cases}$
(Here, r is a constant and θ is the parameter.)

Plot it to see the shape. One arch has range ____ $\leq \theta \leq$ _____.

$$A = \int y\,dx = \int r(1-\cos\theta)r(1-\cos\theta)\,d\theta$$
$$= r^2 \int (1 - 2\cos\theta + \cos^2\theta)\,d\theta$$

$$= 3\pi r^2$$

Area inside a polar curve

For cartesian functions $y = f(x)$, calculate area as $A = \int dA = \int y\,dx$.

What about the area "*inside a curve*" best described as a polar function $r = f(\theta)$?

- We still use $A = \int dA$.
- But the formula for dA is different.

$$A = \int_{\theta=a}^{\theta=b} \frac{1}{2}\left[f(\theta)\right]^2 d\theta = \int_a^b \frac{1}{2} r^2\, d\theta$$

Polar land

How much area is swept out by a little slice?

Example. What is the area inside one loop of the four-leaved rose $r = \cos 2\theta$?

- What are the bounds on θ?
- For which θ does the curve pass through the origin?

$$A = \int_{\theta=\_\_}^{\theta=\_\_} \frac{1}{2}\left[\_\_\_\_\_\_\_\_\right]^2 d\theta$$

Inside yet Outside

Example. Calculate the area inside the curve $r = 3\sin\theta$ and outside the curve $r = \sin\theta + 1$.

First: Draw a picture!

- What are these curves?
- Where do they intersect?

Now calculate: $$\int_{\theta=\_\_\_\_}^{\theta=\_\_\_\_} \Big[\overbrace{\left(\tfrac{1}{2}(3\sin\theta)^2\right)}^{\textit{outside}} - \overbrace{\left(\tfrac{1}{2}(\sin\theta+1)^2\right)}^{\textit{inside}} \Big]\, d\theta$$

$$= \frac{1}{2}\int_{\theta=\_\_}^{\theta=\_\_} \left[9\sin^2\theta - (\sin^2\theta + 2\sin\theta + 1)\right] d\theta$$

$$= \frac{1}{2}\int_{\theta=\_\_}^{\theta=\_\_} \left[8\sin^2\theta - 2\sin\theta - 1\right] d\theta$$

$$= \frac{1}{2}\int_{\theta=\_\_}^{\theta=\_\_} \left[4 - 4\cos 2\theta - 2\sin\theta - 1\right] d\theta$$

$$= \frac{1}{2}\int_{\theta=\_\_}^{\theta=\_\_} \left[3 - 4\cos 2\theta - 2\sin\theta\right] d\theta$$

$$= \left[\frac{3\theta}{2} - \sin 2\theta + \cos\theta\right]_{\pi/6}^{5\pi/6}$$

$$= \frac{3}{2}\left(\frac{5\pi}{6} - \frac{\pi}{6}\right) - \left(\sin\frac{10\pi}{6} - \sin\frac{2\pi}{6}\right) + \left(\cos\frac{5\pi}{6} - \cos\frac{\pi}{6}\right) = \cdots$$

$$= \pi + \sqrt{3} - \sqrt{3} = \pi$$

Arc length of a parametric curve

To find the arc length of a parametric curve, think $L = \int dL$.

How much arc length dL does the curve traverse in one time unit dt?

Draw a right triangle:

To see $dL = \sqrt{dx^2 + dy^2}$.

How are each related to dt?

$$dL = \sqrt{\left[\frac{dx}{dt} \cdot dt\right]^2 + \left[\frac{dy}{dt} \cdot dt\right]^2}$$

$$L = \int_{t=\alpha}^{t=\beta} \sqrt{\left[f'(t)\right]^2 + \left[g'(t)\right]^2}\, dt$$

Example. Find the arc length for the parametric curve $\begin{cases} x = \sin 2t \\ y = \cos 2t \end{cases}$ for $0 \leq t \leq 2\pi$. What do we expect? What is this curve?

$$\int_{t=0}^{t=2\pi} \sqrt{\left[f'(t)\right]^2 + \left[g'(t)\right]^2}\, dt =$$

Arc length of a polar curve

For a polar curve, calculate dL in terms of dr and $d\theta$:

The right triangle shows:

$dL = \sqrt{(r\, d\theta)^2 + (dr)^2}$

Take θ as the parameter to get

$$dL = \sqrt{\left[r\, d\theta\right]^2 + \left[\tfrac{dr}{d\theta}\, d\theta\right]^2} \text{ so } L = \int_{\theta=\alpha}^{\theta=\beta} \sqrt{r^2 + \left(\tfrac{dr}{d\theta}\right)^2}\, d\theta$$

Alternatively, use the parametrization $\begin{cases} x = f(\theta)\cos\theta \\ y = f(\theta)\sin\theta \end{cases}$.

- Then $L = \displaystyle\int_{\theta=\alpha}^{\theta=\beta} \sqrt{\left[\frac{d}{d\theta}\left(f(\theta)\cos\theta\right)\right]^2 + \left[\frac{d}{d\theta}\left(f(\theta)\sin\theta\right)\right]^2}\, d\theta$

. . . take derivatives & do the algebra . . .

$$L = \int_{\theta=\alpha}^{\theta=\beta} \sqrt{\left[f(\theta)\right]^2 + \left[\frac{d}{d\theta}\left(f(\theta)\right)\right]^2}\, d\theta$$