Course Notes

Mathematical Models, Spring 2016

Queens College, Math 245

Prof. Christopher Hanusa

http://qc.edu/~chanusa/courses/245/16/

What is a model?

A model is an object or concept used to represent something else. It converts reality to a form we can comprehend.

- Reality: How to understand the aerodynamics of an airplane?

What is a model?

A model is an object or concept used to represent something else.
It converts reality to a form we can comprehend.

- Reality: How to understand the aerodynamics of an airplane?
  Model: Use a **model airplane** or a **computer simulation**.

What is a model?

A model is an object or concept used to represent something else. It converts reality to a form we can comprehend.

- Reality: How to understand the aerodynamics of an airplane?
  Model: Use a **model airplane** or a **computer simulation**.
- Reality: Politics flows between left-wing and right-wing ideas.

What is a model?

A model is an object or concept used to represent something else. It converts reality to a form we can comprehend.

- Reality: How to understand the aerodynamics of an airplane?
  Model: Use a **model airplane** or a **computer simulation**.
- Reality: Politics flows between left-wing and right-wing ideas.
  Model: Think of public opinion as a **pendulum**.

What is a model?

A model is an object or concept used to represent something else.
It converts reality to a form we can comprehend.

- Reality: How to understand the aerodynamics of an airplane?
  Model: Use a **model airplane** or a **computer simulation**.
- Reality: Politics flows between left-wing and right-wing ideas.
  Model: Think of public opinion as a **pendulum**.

A mathematical model is a model involving mathematical concepts.

What is a model?

A model is an object or concept used to represent something else.
It converts reality to a form we can comprehend.

- Reality: How to understand the aerodynamics of an airplane?
  Model: Use a **model airplane** or a **computer simulation**.
- Reality: Politics flows between left-wing and right-wing ideas.
  Model: Think of public opinion as a **pendulum**.

A mathematical model is a model involving mathematical concepts.

IN THIS CLASS:

We take real-world situations and represent them using mathematics.

- Model the position of a falling object by **function fitting**.
- Model people waiting using a **computer simulation**.
- Model allocating resources using a **system of inequalities**.

What is a model?

A model is an object or concept used to represent something else.
It converts reality to a form we can comprehend.

- Reality: How to understand the aerodynamics of an airplane?
  Model: Use a **model airplane** or a **computer simulation**.
- Reality: Politics flows between left-wing and right-wing ideas.
  Model: Think of public opinion as a **pendulum**.

A mathematical model is a model involving mathematical concepts.

IN THIS CLASS:
We take real-world situations and represent them using mathematics.

- Model the position of a falling object by **function fitting**.
- Model people waiting using a **computer simulation**.
- Model allocating resources using a **system of inequalities**.

Then we must **analyze our models** to determine their applicability.

Why should we model?

As scientists, we want to understand how the world works.

$\longrightarrow$ $\rightsquigarrow$ $\longrightarrow$ observed real-world behavior

Why should we model?

As scientists, we want to understand how the world works.

$\rightarrow$ $\rightsquigarrow$ $\rightarrow$ observed real-world behavior

- What is happening?
- What are the reasons for the behavior?

Why should we model?

As scientists, we want to understand how the world works.

$\longrightarrow$ $\rightsquigarrow$ $\longrightarrow$ observed real-world behavior

- What is happening?
- What are the reasons for the behavior?
- How do we convey that our reasoning is plausible?

Why should we model?

As scientists, we want to understand how the world works.

$\longrightarrow$
$\rightsquigarrow$
$\longrightarrow$

observed real-world behavior

- What is happening? (Observation)
- What are the reasons for the behavior? (Hypothesis)
- How do we convey that our reasoning is plausible? ("proof")

— Use the language of mathematics! —

Steps of the Modeling Process

Goal: Understand what is involved in "mathematical modeling".

First Step: Formulation.

Steps of the Modeling Process

Goal: Understand what is involved in "mathematical modeling".

First Step: Formulation.

- State the question. If the question is vague, make it precise. If the question is too big, subdivide it into manageable parts.

Steps of the Modeling Process

Goal: Understand what is involved in “mathematical modeling”.

First Step: Formulation.

- State the question. If the question is vague, make it precise. If the question is too big, subdivide it into manageable parts.
- Identify factors. Decide which quantities influence the behavior. Determine relationships between the quantities.

Steps of the Modeling Process

Goal: Understand what is involved in “mathematical modeling”.

First Step: Formulation.

- State the question. If the question is vague, make it precise. If the question is too big, subdivide it into manageable parts.
- Identify factors. Decide which quantities influence the behavior. Determine relationships between the quantities.

  [**Important:** *we are introducing* ____________________.]

Steps of the Modeling Process

Goal: Understand what is involved in "mathematical modeling".

First Step: Formulation.

- State the question. If the question is vague, make it precise. If the question is too big, subdivide it into manageable parts.
- Identify factors. Decide which quantities influence the behavior. Determine relationships between the quantities.

  [**Important:** *we are introducing* ______________________.]
- Describe mathematically. Assign each quantity a variable. Represent each relationship with an equation.

Motivating Example: Gravity by Galileo

Example. In Galileo's time, a key question changed from:

Why do objects fall? –to– *How* do objects fall?

Motivating Example: Gravity by Galileo

Example. In Galileo's time, a key question changed from:

Why do objects fall? –to– *How* do objects fall?
(Philosophical question)

Motivating Example: Gravity by Galileo

Example. In Galileo's time, a key question changed from:

Why do objects fall?	–to–	*How* do objects fall?
(Philosophical question)		(Describe a falling object's velocity)

Motivating Example: Gravity by Galileo

Example. In Galileo's time, a key question changed from:

Why do objects fall?	–to–	*How* do objects fall?
(Philosophical question)		(Describe a falling object's velocity)

First Step: Formulation.

- State the question.
- Identify factors.
- Describe mathematically.

Motivating Example: Gravity by Galileo

Example. In Galileo's time, a key question changed from:

Why do objects fall? –to– *How* do objects fall?
(Philosophical question) (Describe a falling object's velocity)

First Step: Formulation.

- State the question.
  What formula describes an object's position as it falls?
- Identify factors.
- Describe mathematically.

Motivating Example: Gravity by Galileo

Example. In Galileo's time, a key question changed from:

Why do objects fall? –to– *How* do objects fall?
(Philosophical question) (Describe a falling object's velocity)

First Step: Formulation.

- State the question. (Precise! What is an answer?)
  What formula describes an object's position as it falls?
- Identify factors.
- Describe mathematically.

Motivating Example: Gravity by Galileo

Example. In Galileo's time, a key question changed from:

Why do objects fall? –to– *How* do objects fall?
(Philosophical question) (Describe a falling object's velocity)

First Step: Formulation.

- State the question. (Precise! What is an answer?)
  What formula describes an object's position as it falls?
- Identify factors. Galileo chose only distance, time, and velocity.
  Other variables?: ______________________________
- Describe mathematically.

Motivating Example: Gravity by Galileo

Example. In Galileo's time, a key question changed from:

Why do objects fall? –to– *How* do objects fall?
(Philosophical question) (Describe a falling object's velocity)

First Step: Formulation.

- State the question. (Precise! What is an answer?)
  What formula describes an object's position as it falls?
- Identify factors. Galileo chose only distance, time, and velocity.
  Other variables?: ______________________________

Simplifying Assumption: Velocity is proportional to the distance fallen.

- Describe mathematically.

Motivating Example: Gravity by Galileo

Example. In Galileo's time, a key question changed from:

Why do objects fall? –to– *How* do objects fall?
(Philosophical question) (Describe a falling object's velocity)

First Step: Formulation.

- State the question. (Precise! What is an answer?)
  What formula describes an object's position as it falls?
- Identify factors. Galileo chose only distance, time, and velocity.
  Other variables?: ________________________________

Simplifying Assumption: Velocity is proportional to the distance fallen.

- Describe mathematically.
  Assign variables. Call distance x, time t, and velocity v.

Motivating Example: Gravity by Galileo

Example. In Galileo's time, a key question changed from:

Why do objects fall? –to– *How* do objects fall?
(Philosophical question) (Describe a falling object's velocity)

First Step: Formulation.

- State the question. (Precise! What is an answer?)
  What formula describes an object's position as it falls?
- Identify factors. Galileo chose only distance, time, and velocity.
  Other variables?: ______________________

Simplifying Assumption: Velocity is proportional to the distance fallen.

- Describe mathematically.
  Assign variables. Call distance x, time t, and velocity v.
  Then relationships give equations:
  Velocity and distance are related:

Motivating Example: Gravity by Galileo

Example. In Galileo's time, a key question changed from:

Why do objects fall? –to– *How* do objects fall?
(Philosophical question) (Describe a falling object's velocity)

First Step: Formulation.

- State the question. (Precise! What is an answer?)
  What formula describes an object's position as it falls?
- Identify factors. Galileo chose only distance, time, and velocity.
  Other variables?: ______________________________

Simplifying Assumption: Velocity is proportional to the distance fallen.

- Describe mathematically.
  Assign variables. Call distance x, time t, and velocity v.
  Then relationships give equations:
  Velocity and distance are related: $v = \frac{dx}{dt}$.

Motivating Example: Gravity by Galileo

Example. In Galileo's time, a key question changed from:

Why do objects fall? –to– *How* do objects fall?
(Philosophical question) (Describe a falling object's velocity)

First Step: Formulation.

- State the question. (Precise! What is an answer?)
  What formula describes an object's position as it falls?
- Identify factors. Galileo chose only distance, time, and velocity.
  Other variables?: ________________________________

Simplifying Assumption: Velocity is proportional to the distance fallen.

- Describe mathematically.
  Assign variables. Call distance x, time t, and velocity v.
  Then relationships give equations:
  Velocity and distance are related: $v = \frac{dx}{dt}$.
  And *proportional* means $v = ax$ for some constant a. (Goal?)

Steps of the Modeling Process

- After the formulation step, we have variables and equations.

Steps of the Modeling Process

- After the formulation step, we have variables and equations.
- Do some analysis to develop **mathematical conclusions**.

Steps of the Modeling Process

- After the formulation step, we have variables and equations.
- Do some analysis to develop **mathematical conclusions**.

Second Step: Mathematical Manipulation.

This may entail one or more of:

Steps of the Modeling Process

- After the formulation step, we have variables and equations.
- Do some analysis to develop **mathematical conclusions**.

Second Step: Mathematical Manipulation.

This may entail one or more of:

- Calculations

Steps of the Modeling Process

- After the formulation step, we have variables and equations.
- Do some analysis to develop **mathematical conclusions**.

Second Step: Mathematical Manipulation.

This may entail one or more of:

- Calculations
- Proving a theorem

Steps of the Modeling Process

- After the formulation step, we have variables and equations.
- Do some analysis to develop **mathematical conclusions**.

Second Step: Mathematical Manipulation.

This may entail one or more of:

- Calculations
- Solving an equation
- Proving a theorem

Steps of the Modeling Process

- After the formulation step, we have variables and equations.
- Do some analysis to develop **mathematical conclusions**.

Second Step: Mathematical Manipulation.

This may entail one or more of:

- Calculations
- Solving an equation
- Proving a theorem
- Other...

Steps of the Modeling Process

- After the formulation step, we have variables and equations.
- Do some analysis to develop **mathematical conclusions**.

Second Step: Mathematical Manipulation.

This may entail one or more of:

- Calculations
- Solving an equation
- Proving a theorem
- Other...

In our gravity example,

We have both $v = \frac{dx}{dt}$ and $v = ax$. Set them equal.

Steps of the Modeling Process

- After the formulation step, we have variables and equations.
- Do some analysis to develop **mathematical conclusions**.

Second Step: Mathematical Manipulation.

This may entail one or more of:

- Calculations
- Solving an equation
- Proving a theorem
- Other...

In our gravity example,

We have both $v = \frac{dx}{dt}$ and $v = ax$. Set them equal.

This gives the (differential) equation: $\frac{dx}{dt} = ax$.

Steps of the Modeling Process

- After the formulation step, we have variables and equations.
- Do some analysis to develop **mathematical conclusions**.

Second Step: Mathematical Manipulation.

This may entail one or more of:

- Calculations
- Solving an equation
- Proving a theorem
- Other...

In our gravity example,

We have both $v = \frac{dx}{dt}$ and $v = ax$. Set them equal.

This gives the (differential) equation: $\frac{dx}{dt} = ax$.

Solving gives the equation $x(t) = ke^{at}$ for constants a and k.

Steps of the Modeling Process

- After the formulation step, we have variables and equations.
- Do some analysis to develop **mathematical conclusions**.

Second Step: Mathematical Manipulation.

This may entail one or more of:

- Calculations
- Solving an equation
- Proving a theorem
- Other...

In our gravity example,

We have both $v = \frac{dx}{dt}$ and $v = ax$. Set them equal.

This gives the (differential) equation: $\frac{dx}{dt} = ax$.

Solving gives the equation $x(t) = ke^{at}$ for constants a and k.

Something is not quite right...

Steps of the Modeling Process

We have a mathematical conclusion, but does it give a “right answer”?

Steps of the Modeling Process

We have a mathematical conclusion, but does it give a “right answer”?

The *most important* step of the modeling process is:

Third Step: Evaluation.

Steps of the Modeling Process

We have a mathematical conclusion, but does it give a "right answer"?

The *most important* step of the modeling process is:

Third Step: Evaluation.

Translate the results back to the real-world situation and ask questions:

Steps of the Modeling Process

We have a mathematical conclusion, but does it give a “right answer”?

The *most important* step of the modeling process is:

Third Step: Evaluation.

Translate the results back to the real-world situation and ask questions:

- Has the model explained the real-world observations?
- Are the answers we found accurate enough?
- Were our assumptions good assumptions?
- What are the strengths and weaknesses of our model?
- Did we make any mistakes in our mathematical manipulations?

Steps of the Modeling Process

We have a mathematical conclusion, but does it give a "right answer"?

The *most important* step of the modeling process is:

Third Step: Evaluation.

Translate the results back to the real-world situation and ask questions:

- Has the model explained the real-world observations?
- Are the answers we found accurate enough?
- Were our assumptions good assumptions?
- What are the strengths and weaknesses of our model?
- Did we make any mistakes in our mathematical manipulations?

If there are any problems,

- **Go back** to the First Step, Formulation.
- Change your assumptions!
- Start the modeling process over.

Motivating Example: Gravity by Galileo

Third Step: Evaluation.

Our mathematical calculations imply that the position of a falling object is $x(t) = ke^{at}$.

Motivating Example: Gravity by Galileo

Third Step: Evaluation.

Our mathematical calculations imply that the position of a falling object is $x(t) = ke^{at}$.

In our real-world situation, we can set initial position to be 0. Mathematically, $x(0) = 0$.

Motivating Example: Gravity by Galileo

Third Step: Evaluation.

Our mathematical calculations imply that the position of a falling object is $x(t) = ke^{at}$.

In our real-world situation, we can set initial position to be 0. Mathematically, $x(0) = 0$.

This lets us solve for k in our equation:
$0 = x(0) = ke^{a0} = ke^{0} = k$.

Motivating Example: Gravity by Galileo

Third Step: Evaluation.

Our mathematical calculations imply that the position of a falling object is $x(t) = ke^{at}$.

In our real-world situation, we can set initial position to be 0. Mathematically, $x(0) = 0$.

This lets us solve for k in our equation:
$0 = x(0) = ke^{a0} = ke^{0} = k$.

So $k = 0$. Plugging into our equation implies

Motivating Example: Gravity by Galileo

Third Step: Evaluation.

Our mathematical calculations imply that the position of a falling object is $x(t) = ke^{at}$.

In our real-world situation, we can set initial position to be 0. Mathematically, $x(0) = 0$.

This lets us solve for k in our equation:
$0 = x(0) = ke^{a0} = ke^{0} = k$.

So $k = 0$. Plugging into our equation implies $x(t) = 0$.

Motivating Example: Gravity by Galileo

Third Step: Evaluation.

Our mathematical calculations imply that the position of a falling object is $x(t) = ke^{at}$.

In our real-world situation, we can set initial position to be 0. Mathematically, $x(0) = 0$.

This lets us solve for k in our equation:
$0 = x(0) = ke^{a0} = ke^{0} = k$.

So $k = 0$. Plugging into our equation implies $x(t) = 0$.

In words, this means that our object stays at rest for all t.

Motivating Example: Gravity by Galileo

Third Step: Evaluation.

Our mathematical calculations imply that the position of a falling object is $x(t) = ke^{at}$.

In our real-world situation, we can set initial position to be 0. Mathematically, $x(0) = 0$.

This lets us solve for k in our equation:
$0 = x(0) = ke^{a0} = ke^{0} = k$.

So $k = 0$. Plugging into our equation implies $x(t) = 0$.

In words, this means that our object stays at rest for all t.

EPIC FAIL!

Motivating Example: Gravity by Galileo

Third Step: Evaluation.

Our mathematical calculations imply that the position of a falling object is $x(t) = ke^{at}$.

In our real-world situation, we can set initial position to be 0. Mathematically, $x(0) = 0$.

This lets us solve for k in our equation:
$0 = x(0) = ke^{a0} = ke^{0} = k$.

So $k = 0$. Plugging into our equation implies $x(t) = 0$.

In words, this means that our object stays at rest for all t.

EPIC FAIL!

Perhaps the proportionality assumption is incorrect?

Motivating Example: Gravity by Galileo

First Step: Formulation.

Alternate assumption: The velocity is proportional to the time it has been falling. In particular, the velocity increases by 32 ft/sec.

Motivating Example: Gravity by Galileo

First Step: Formulation.

Alternate assumption: The velocity is proportional to the time it has been falling. In particular, the velocity increases by 32 ft/sec.

Mathematically, we have the equations $v = 32t$ and $v = \frac{dx}{dt}$.

Motivating Example: Gravity by Galileo

First Step: Formulation.

Alternate assumption: The velocity is proportional to the time it has been falling. In particular, the velocity increases by 32 ft/sec.

Mathematically, we have the equations $v = 32t$ and $v = \frac{dx}{dt}$.

Second Step: Mathematical Manipulation.

Integrating gives $x(t) = 16t^2 + C$.

Motivating Example: Gravity by Galileo

First Step: Formulation.

Alternate assumption: The velocity is proportional to the time it has been falling. In particular, the velocity increases by 32 ft/sec.

Mathematically, we have the equations $v = 32t$ and $v = \frac{dx}{dt}$.

Second Step: Mathematical Manipulation.

Integrating gives $x(t) = 16t^2 + C$.

Since $x(0) = 0$ we can find $C = 0$.

Motivating Example: Gravity by Galileo

First Step: Formulation.

Alternate assumption: The velocity is proportional to the time it has been falling. In particular, the velocity increases by 32 ft/sec.

Mathematically, we have the equations $v = 32t$ and $v = \frac{dx}{dt}$.

Second Step: Mathematical Manipulation.

Integrating gives $x(t) = 16t^2 + C$.

Since $x(0) = 0$ we can find $C = 0$.

Therefore an object falling from rest has position $x(t) = 16t^2$.

Motivating Example: Gravity by Galileo

First Step: Formulation.

Alternate assumption: The velocity is proportional to the time it has been falling. In particular, the velocity increases by 32 ft/sec.

Mathematically, we have the equations $v = 32t$ and $v = \frac{dx}{dt}$.

Second Step: Mathematical Manipulation.

Integrating gives $x(t) = 16t^2 + C$.

Since $x(0) = 0$ we can find $C = 0$.

Therefore an object falling from rest has position $x(t) = 16t^2$.

Third Step: Evaluation.

This function agrees well with observations in many instances.

Motivating Example: Gravity by Galileo

First Step: Formulation.

Alternate assumption: The velocity is proportional to the time it has been falling. In particular, the velocity increases by 32 ft/sec.

Mathematically, we have the equations $v = 32t$ and $v = \frac{dx}{dt}$.

Second Step: Mathematical Manipulation.

Integrating gives $x(t) = 16t^2 + C$.

Since $x(0) = 0$ we can find $C = 0$.

Therefore an object falling from rest has position $x(t) = 16t^2$.

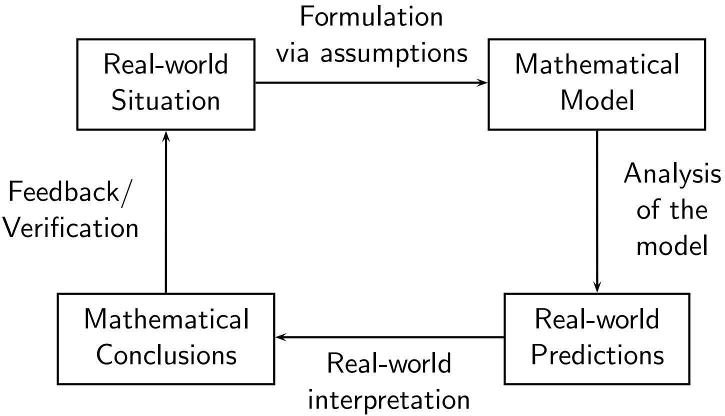
Third Step: Evaluation.

This function agrees well with observations in many instances.

(Although not all!)

The Modeling Process

This chart summarizes the modeling process.



To do well in this class:

- **Come to class prepared.**
  - Print out and read over course notes.
  - Read assigned sections before class.
- **Form good study groups.**
  - Discuss homework and classwork.
  - Final project is a group project.
  - You will depend on this group.
- **Put in the time.**
  - Three credits = (at least) nine hours / week out of class.
  - Homework stresses key concepts from class; learning takes time.
- **Stay in contact.**
  - If you are confused, ask questions (in class and out).
  - Don't fall behind in coursework or project.
  - I need to understand your concerns.

Homework posted online; Email me by Monday.

Choosing a problem statement.

Group Activity. Arrange yourselves into groups of four or five people, with people you **don't know**.

- Introduce yourself. (your name, where you're from, your major)
- Fill out **the front of** your notecard:
  - Write your name. (Stylize if you wish.)
  - Write a few words related to your name.
  - *Draw* something in the remaining space.
- Discuss with your groupmates why you wrote what you wrote.
- Exchange contact information. (phone / email / other)

- Work in your group on the worksheet.