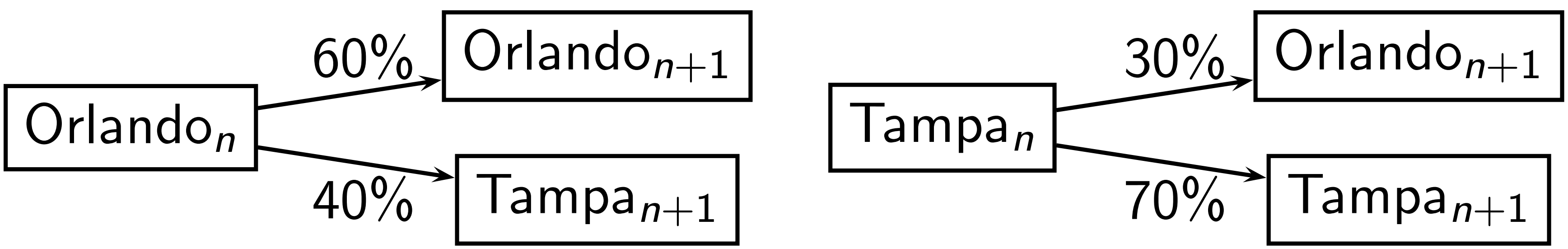
Markov Chains

A **Markov chain** is a sequence of trials of an experiment in which

- The outcome of each experiment is from a set of discrete states.
- Each outcome depends only on the present state: There is a *fixed* probability $a_{i,j}$ of transitioning from state j to state i.

Example. Suppose you run a rental company based in Orlando and Tampa, Florida. People often drive between the cities; cars can be picked up and dropped off in either city. Suppose that historically, when a car is picked up in a city at time n,



What distribution of cars can the company expect in the long run?

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We will model this situation with a Markov Chain.

Use the historical data to form the transition matrix A.
The transition probability from Orlando at time n to Orlando at time $n+1$ is:

FROM:

$$\textbf{TO}: \begin{matrix} & \text{Or} & \text{Tm} \\ \text{Or} & & \\ \text{Tm} & & \end{matrix} \quad \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} = A,$$

- Let o_n be the probability that a car is in Orlando on day n
- Let t_n be the probability that a car is in Tampa on day n.

We can represent the distribution of cars at time n with the vector

$\vec{\mathbf{x}}_n = \begin{bmatrix} o_n \\ t_n \end{bmatrix}$. And so, $\vec{\mathbf{x}}_{n+1} = \begin{bmatrix} o_{n+1} \\ t_{n+1} \end{bmatrix} = A \cdot \begin{bmatrix} o_n \\ t_n \end{bmatrix} = A\vec{\mathbf{x}}_n$.

Given an initial distribution $\vec{\mathbf{x}}_0 = \begin{bmatrix} o_0 \\ t_0 \end{bmatrix}$,

the expected distribution of cars at time n is $\vec{\mathbf{x}}_n =$ ____.

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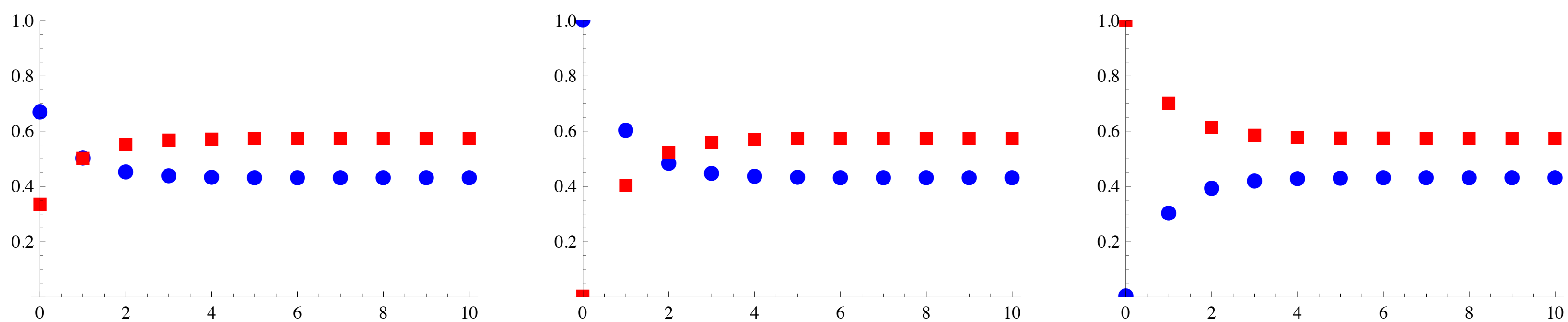
For example, if they company starts off with twice as many cars in Orlando as in Tampa, then $\vec{\mathbf{x}}_0 = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$, so we expect

$$\vec{\mathbf{x}}_1 = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}.$$

$$\vec{\mathbf{x}}_2 = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} \begin{bmatrix} \quad \\ \quad \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}.$$



Notice: Same equilibrium distribution each time!
Notice: The equilibrium is approached quickly!

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Definition: Given a Markov Chain with transition matrix A, an **equilibrium distribution** is a vector $\vec{\mathbf{x}}_{eq}$ that satisfies $A\vec{\mathbf{x}}_{eq} = \vec{\mathbf{x}}_{eq}$.
[Linear Algebra: $\vec{\mathbf{x}}_{eq}$ *is an eigenvector corresponding to* $\lambda = 1$.]

In our example, the equilibrium distribution satisfies

$$\begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} \begin{bmatrix} o_{eq} \\ t_{eq} \end{bmatrix} = \begin{bmatrix} o_{eq} \\ t_{eq} \end{bmatrix}.$$

So solve: $0.6o_{eq} + 0.3t_{eq} = o_{eq}$ and $0.4o_{eq} + 0.7t_{eq} = t_{eq}$.
Both equations reduce to $0.3t_{eq} = 0.4o_{eq}$, so $o_{eq} = \frac{3}{4}t_{eq}$.

Conclusion: If the company has 7000 cars in all, they would expect that in the long run, ________________________________

In Markov Chains: ★ The sum of the entries in every column of A is 1, because the total probability of transitioning **from** state i is 1.

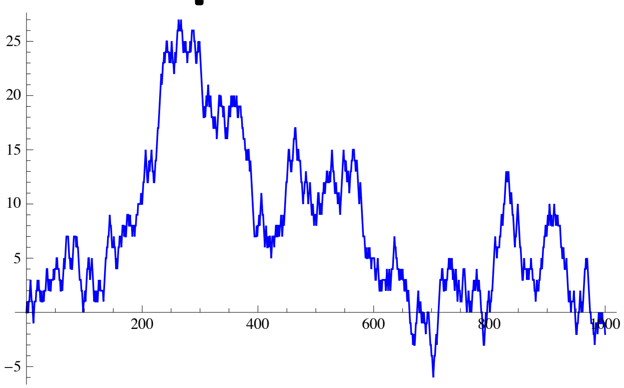
★ There is no general rule for what the row sum will be.

Random Walk

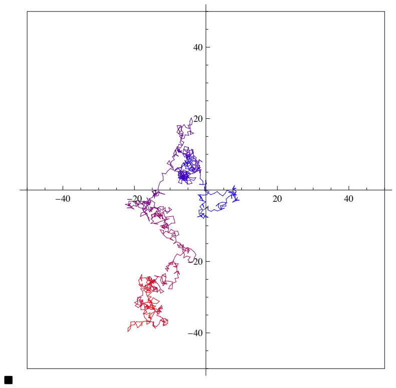
A **random walk** is a sequence of steps, where each step is generated randomly and depends only on its current position.

Random walks can be thought of as a special type of Markov chain.

Stock prices



Movement of a molecule in liquid (Wiener process)



Genetic drift

Diffusion of populations

Polymers

Shuffling of a deck of cards.
Each state is one of the $n!$ permutations of the n cards.
We transition from one state to another by some rule. Perhaps:

- Moving a random card to a new position.
- Choosing a pair of random cards and exchanging them.

Simple random walk

A drunk in a bar. A bar patron has had a little too much to drink and it's about time to leave the bar. There is an exit directly to his right and an exit three steps away to his left. The drunk stumbles randomly one step to the left or one step to the right with equal probability.

What is the probability that the drunk leaves via the right door?

What is the transition matrix for this random walk?

What is an equilibrium solution for this random walk?

Gambler's Ruin

Win or go home broke! A gambler starts with \$500 and makes \$1 bets, winning each with probability p.

The gambler stays until she has made \$100 profit or goes broke.

Question. What is the probability that she goes home a winner?

This depends on p. For roulette: $p = 18/38 \approx 47.3\%$:

The probability of achieving \$600 before going broke is

We can model this with a random walk.

- One state per dollar value.
- Probability $d \to d+1$: _____ Probability $d \to d-1$: _____

Theorem. The probability that a gambler starting with n dollars acheives T dollars before going broke is

$$\left[\left(\frac{1-p}{p}\right)^n - 1\right] \Big/ \left[\left(\frac{1-p}{p}\right)^T - 1\right].$$

Color mixing game

Let's play an interactive Markov chain game.

- Choose a color. (Red, Orange, Yellow, Green, Blue, Purple)
- Record the distribution.
- Do some Markov mixing.
  - Find a random partner. Announce your colors.
  - Randomly decide whose color will prevail.
    (Coin flip or Rock Paper Scissors)
  - Both players now take the winning color.
  - Repeat many times!
- Record the distribution at multiple times during the experiment.

What do we expect to occur?

Stand up and make some space to move around.