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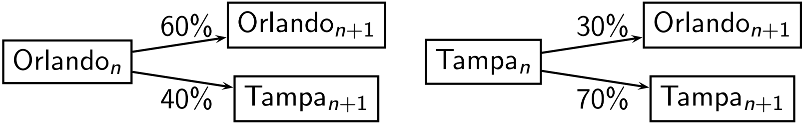
Example. Suppose you run a rental company based in Orlando and Tampa, Florida. People often drive between the cities; cars can be picked up and dropped off in either city. Suppose that historically, when a car is picked up in a city at time n,



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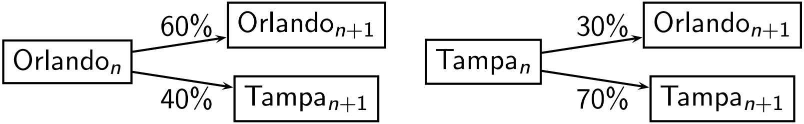

What distribution of cars can the company expect in the long run?

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We will model this situation with a Markov Chain.

Use the historical data to form the transition matrix A.
The transition probability from Orlando at time n to Orlando at time $n+1$ is:

FROM:

$$\textbf{TO:}\quad \begin{array}{c} \\ \text{Or} \\ \text{Tm} \end{array} \begin{array}{c} \begin{array}{cc} \text{Or} & \text{Tm} \end{array} \\ \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \end{array} = A,$$

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Given an initial distribution $\vec{\mathbf{x}}_0 = \begin{bmatrix} o_0 \\ t_0 \end{bmatrix}$,

the expected distribution of cars at time n is $\vec{\mathbf{x}}_n = $ ____.

Markov Chains

For example, if they company starts off with twice as many cars in Orlando as in Tampa, then $\vec{\mathbf{x}}_0 = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$, so we expect

$$\vec{\mathbf{x}}_1 = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}.$$

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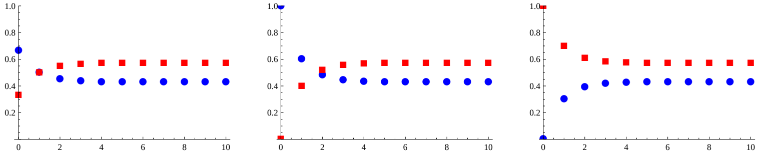

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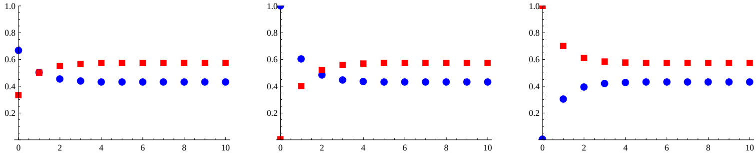

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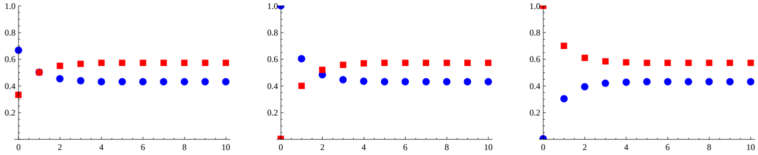

Notice: Same equilibrium distribution each time!
Notice: The equilibrium is approached quickly!

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★ There is no general rule for what the row sum will be.

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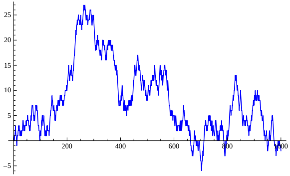

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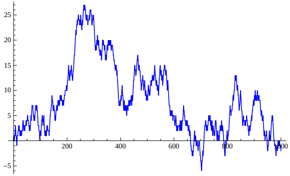
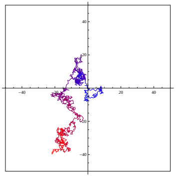


Movement of a molecule in liquid (Wiener process)



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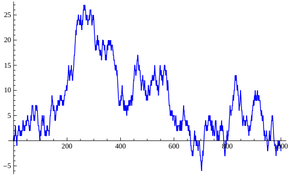

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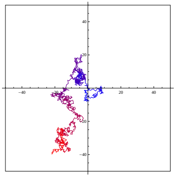

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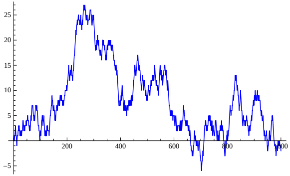

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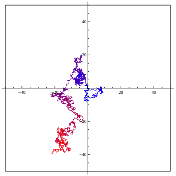

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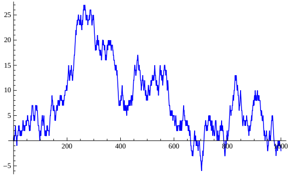

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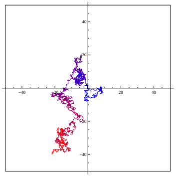

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We transition from one state to another by some rule. Perhaps:

- Moving a random card to a new position.
- Choosing a pair of random cards and exchanging them.

Simple random walk

A drunk in a bar. A bar patron has had a little too much to drink and it's about time to leave the bar. There is an exit directly to his right and an exit three steps away to his left. The drunk stumbles randomly one step to the left or one step to the right with equal probability.

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What is an equilibrium solution for this random walk?

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Win or go home broke! A gambler starts with \$500 and makes \$1 bets, winning each with probability p.

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Theorem. The probability that a gambler starting with n dollars acheives T dollars before going broke is

$$\left[\left(\frac{1-p}{p}\right)^n - 1\right] \bigg/ \left[\left(\frac{1-p}{p}\right)^T - 1\right].$$

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- Choose a color. (Red, Orange, Yellow, Green, Blue, Purple)
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Stand up and make some space to move around.