Evaluation of Mathematical Models

In what ways can a model be "good"? A model can be...

- Accurate
 - Is the output of the model very near to correct?
- Descriptively Realistic
 - Is the model based on assumptions which are correct?
- Precise
 - ▶ Are the predictors of the model definite numbers?
- Robust
 - Is the model relatively immune to errors in the input data?
- General
 - Does the model apply to a wide variety of situations?
- ▶ Fruitful
 - Are the conclusions useful?
 - Does the model inspire other good models?

Precise Robust General

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Model Assumption 2: One of every two is enrolled in college.

If next year there are projected to be 11,000,000 18-22 year olds, we would estimate the college population to be of size E =

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Question: Is this model descriptively realistic?

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Example. A more descriptively realistic model would incorporate other age groups. Replace Assumptions 1 and 2 by:

Model Assumption 3: College students are either:

- ▶ 18-22 (P_a of these)
- \triangleright 23 or older (P_b of these)
- ▶ 17 or younger (P_c of these)

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Model Assumption 4: The enrolled percentages for each age range is:

- ▶ 30% for people aged 18–22
- ▶ 3% for people aged 23 or older
- ▶ 1% for people aged 17 or younger

We would estimate the college population to be of size

$$E = 0.3P_a + 0.03P_b + 0.01P_b$$
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Accurate Real **Precise** Robust General Fruitful

Fruitful

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a range of numbers a set of functions etc.

Circle one: The enrollment models are precise imprecise. Why?

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A model is $\begin{cases} \textbf{precise} \text{ if the prediction is } \begin{cases} a \text{ definite number} \\ a \text{ definite function} \\ \text{etc.} \end{cases} \\ \textbf{imprecise} \text{ if the prediction is } \begin{cases} a \text{ definite number} \\ a \text{ definite function} \\ \text{etc.} \end{cases}$

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Keep Assumption 1: Each college student is in 18–22 year old range.

Revise Assumption 2^* : The percentage of 18–22 year olds in college is between 46% and 50%. (Historically)

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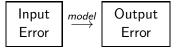
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This model is imprecise, but perhaps more helpful than the precise answer from before.

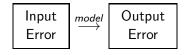
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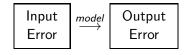
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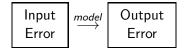


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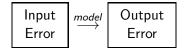
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Make sure we understand: What does 10% error mean?

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This highlights the principle of "Error In equals Error Out"

Robust General

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Then comparing the true enrollment to the estimated enrollment E':

$$E = 0.3(10) + 0.03(90) + 0.01(50) = 6.2$$

 $E' = 0.3(11) + 0.03(99) + 0.01(55) = 6.82$

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Percentage error: $\frac{6.22-6.2}{6.2} = \frac{.02}{6.2} = 0.3\%$.

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Precise Robust **General**

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It is complicated to estimate total enrollment using this model. This model is more general because it applies to individual colleges.

Fruitfulness

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Example. How many automobiles would be junked in a given year?

- ► Cars play the role of people.
- ▶ Partitioning by age of cars gives better results

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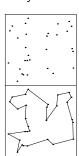
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Example. The Traveling Salesman Problem

TSP: Given a home location and a set of places to visit, find the shortest path that starts and ends at home and visits each of the places along the way.



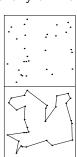
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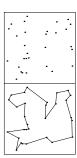
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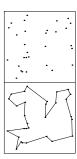
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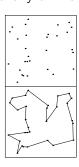
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- ▶ If you visit the same places every day, run the expensive model once initially in order to save money in the long run.
- ▶ If you visit different places every day, run the inexpensive algorithm daily. (Unless you're UPS or FedEx.)