Deterministic versus Probabilistic

Two differing views of modeling:

Deterministic: All data is known beforehand

- Once the system starts, you know exactly what is going to happen.
- Example. Predicting the amount of money in a bank account.
  - If you know the initial deposit, and the interest rate, then:
  - You can determine the amount in the account after one year.

Probabilistic: Element of chance is involved

- You know the likelihood that something will happen, but you don't know when it will happen.
- Example. Roll a die until it comes up '5'.
  - Know that in each roll, a '5' will come up with probability $1/6$.
  - Don't know exactly when, but we can predict well.

Question: Is a game of Candy Land deterministic or probabilistic?

Basic Probability

Definition: An **experiment** is any process whose outcome is uncertain.

Definition: The set of all possible outcomes of an experiment is called the **sample space**, denoted X (or S).

Definition: The **probability** of x, denoted $p(x)$, is a number between 0 and 1 that measures its likelihood of occurring.

Example. Rolling a die is an experiment; the sample space is $\{\_\_\_\_\_\_\_\_\_\_\}$. The individual probabilities are all $p(i) =$ _____.

Definition: An **event** E is something that can happen.
(In other words, it is a subset of the sample space: $E \subset X$.)

Definition: The **probability** of an event E, $p(E)$, is the sum of the probabilities of the outcomes making up the event.

Example. The roll of the die ... [is '5'] or [is odd] or [is prime] ...

Example. $p(E_1) =$________, $p(E_2) =$________, $p(E_3) =$________.

Determining Probabilities

Three methods for modeling the probability of an occurrence:

- **Relative frequency method:** Repeat an experiment many times; assign as the probability the fraction $\frac{\text{occurrences}}{\#\text{ experiments run}}$.
  Example. Hit a bulls-eye 17 times out of 100; set the probability of hitting a bulls-eye to be $p(\text{bulls-eye}) = 0.17$.

- **Equal probability method:** Assume all outcomes have equal probability; assign as the probability $\frac{1}{\#\text{ of possible outcomes}}$.
  Example. Each side of a dodecahedral die is equally likely to appear; decide to set $p(1) = \frac{1}{12}$.

- **Subjective guess method:**
  If neither method above applies, give it your best guess.
  Example. How likely is it that your friend will come to a party?

Independent Events

Definition: Two events are **independent** if the probabilities of occurrence do not depend on one another.

Example. Roll a Red die and roll a Blue die.

- **Event 1:** Blue die shows a 1. **Event 2:** Red die shows a 6. These events are independent.
- **Event 1:** Blue die shows a 1. **Event 2:** Blue die shows a 6. These events are dependent.

Example. Shuffle a deck of 52 cards. Pick two cards.

- **Event 1:** First card is A♡. **Event 2:** Second card is K♠. These events are ____________.

Example. You wake up and don't know what day it is.

- **Event 1:** Today is a weekday. E_1 vs. E_2
- **Event 2:** Today is cloudy. E_2 vs. E_3
- **Event 3:** Today is Modeling day. E_1 vs. E_3

Independent Events

- When events E_1 (in X_1) and E_2 (in X_2) are *independent* events,

$$p(E_1 \text{ AND } E_2) = p(E_1)p(E_2).$$

Example. What is the probability
that today is a cloudy weekday?

- For events E_1 (in X_1) and E_2 (in X_2),

$$p(E_1 \text{ OR } E_2) = 1 - p((\text{NOT } E_1) \text{ AND } (\text{NOT } E_2))$$
$$=$$
$$\text{(when indep.)} = p(E_1) + p(E_2) - p(E_1)p(E_2)$$

Proof: Venn diagram / rectangle

Example. What is the probability that you roll a blue 1 OR a red 6?

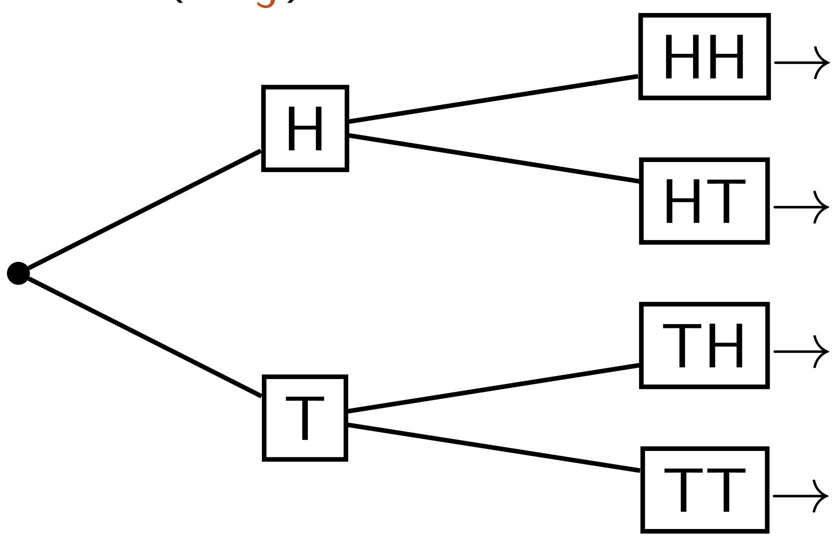
This does not work with *dependent* events.

Decision Trees

Definition: A **multistage** experiment is one in which each stage is a simpler experiment. They can be represented using a **tree diagram**.

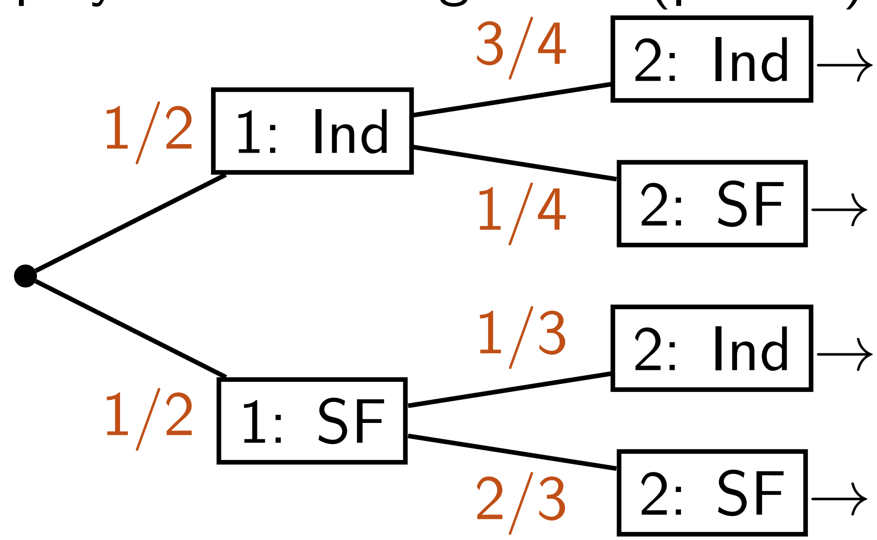
Each branch of the tree represents one outcome x of that level's experiment, and is labeled by $p(x)$.

Example. Flipping a biased coin twice (H:$\frac{2}{3}$).



Independent or dependent?

Example. Indiana and SF State U. play two soccer games. (p. 382)



Independent or dependent?

Expected value / mean

"Even with the randomness, what do you expect to happen?"

Suppose that each outcome x in a sample space has a number $r(x)$ attached to it. (**Examples:** number of pips on a die, amount of money you win on a bet, inches of precipitation falling)

This function r is called a **random variable**.

Definition: The **expected value** or **mean** of a random variable is the sum of the numbers weighted by their probabilities. Mathematically,

$$\mu = \mathbb{E}[X] = p(x_1)r(x_1) + p(x_2)r(x_2) + \cdots + p(x_n)r(x_n).$$

Idea: With probability $p(x_1)$, there is a contribution of $r(x_1)$, etc.
Example. How many heads would you expect on average when flipping a biased coin twice?

Example. How many wins do you expect Indiana to have?

Question: What is the **interpretation** of this number $\mathbb{E}[X]$?

Expected value / mean

When two random variables are on two **independent** experiments, the expected value operation behaves nicely:

$$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y] \quad \text{and} \quad \mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y].$$

Example. We throw a red die and a blue die. What is the expected value of the sum of the dice and the product of the dice?

$_b+^r$	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

$_b*^r$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	8	10	12
3	3	6	9	12	15	18
4	4	8	12	16	20	24
5	5	10	15	20	25	30
6	6	12	18	24	30	36

$\mathbb{E}[X+Y] =$

$\mathbb{E}[XY] =$

Component Reliability

Many systems consist of components pieced together.

Definition: The **reliability** of a system is its probability of success.

To calculate system reliability, first determine how reliable each component is; then apply rules from probability.

Example. Launch the space shuttle into space with a three-stage rocket.

$$\boxed{\text{Stage 1}} \rightarrow \boxed{\text{Stage 2}} \rightarrow \boxed{\text{Stage 3}}$$

⋆ In order for the rocket to launch, ______________________ ⋆

Let $R_1 = 90\%$, $R_2 = 95\%$, $R_3 = 96\%$ be the reliabilities of Stages 1–3.

$p(\text{system success}) =$
$p(\text{S1 success AND S2 success AND S3 success})$

Component Reliability

Example. Communicating with the space shuttle.

There are two independent methods in which earth can communicate with the space shuttle

- A microwave radio with reliability $R_1 = 0.95$
- An FM radio, with reliability $R_2 = 0.96$.

⋆ In order to be able to communicate with the shuttle, ______________________________.

$p(\text{system success}) = p(\text{MW radio success OR FM radio success})$