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- **Formulation.** Suppose the problem has been properly formulated.
  - Problem statement is precise and clear.
  - Dependent variable(s) and independent variable determined.
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▶ **Evaluation.** Does this function fit the data well?
  ★ For a real world question, there is more evaluation to do.
Functions you should recognize on sight

What are these functions?
What is the most general equation of each type?
Springs and Elongations

**Example:** Modeling Spring Elongation

Take your favorite spring. Attach different masses. How much does it stretch from rest? [Its elongation.]
Springs and Elongations

Example: Modeling Spring Elongation

Take your favorite spring. Attach different masses. How much does it stretch from rest? [Its elongation.]

When we plot the data, we get the following scatterplot.

<table>
<thead>
<tr>
<th>Mass (x)</th>
<th>Elongation (e)</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>1.000</td>
</tr>
<tr>
<td>75</td>
<td>1.875</td>
</tr>
<tr>
<td>100</td>
<td>2.750</td>
</tr>
<tr>
<td>150</td>
<td>4.375</td>
</tr>
<tr>
<td>200</td>
<td>6.500</td>
</tr>
<tr>
<td>250</td>
<td>8.000</td>
</tr>
<tr>
<td>300</td>
<td>9.750</td>
</tr>
<tr>
<td>350</td>
<td>11.250</td>
</tr>
<tr>
<td>400</td>
<td>12.750</td>
</tr>
<tr>
<td>450</td>
<td>14.250</td>
</tr>
<tr>
<td>500</td>
<td>15.750</td>
</tr>
<tr>
<td>550</td>
<td>17.250</td>
</tr>
</tbody>
</table>

What do you notice?
Proportionality

When data seems to lie on a line through the origin, we expect the two variables to be proportional; in this case, $y = kx$ for some constant $k$. 
Proportionality

When data seems to lie on a line through the origin, we expect the two variables to be proportional; in this case, \( y = kx \) for some constant \( k \).

We need to find this constant of proportionality \( k \).
Proportionality

When data seems to lie on a line through the origin, we expect the two variables to be **proportional**; in this case, \( y = kx \) for some constant \( k \). We need to find this **constant of proportionality** \( k \).

So: Estimate the slope of the line. **How?**

1. Guesstimating
Proportionality

When data seems to lie on a line through the origin, we expect the two variables to be **proportional**; in this case, \( y = kx \) for some constant \( k \).

We need to find this **constant of proportionality** \( k \).

So: Estimate the slope of the line. **How?**

1. Guesstimating

2. Mathematically: **Linear Regression / Least Squares**
   (For another day)
Fitting Gravity Data

Example. Modeling the dropping of a golf ball

Source: practicalphysics.org
Example. Modeling the dropping of a golf ball

Let’s use an experiment to test the gravity model from last time.

Source:
practicalphysics.org
Fitting Gravity Data

Example. Modeling the dropping of a golf ball

Let’s use an experiment to test the gravity model from last time.

Use a camera to record the position every tenth of a second.

Source:
practicalphysics.org
Fitting Gravity Data

Example. Modeling the dropping of a golf ball

Let’s use an experiment to test the gravity model from last time.

Use a camera to record the position every tenth of a second.

Data would be similar to the table →

<table>
<thead>
<tr>
<th>t</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.00</td>
</tr>
<tr>
<td>0.1</td>
<td>0.25</td>
</tr>
<tr>
<td>0.2</td>
<td>0.75</td>
</tr>
<tr>
<td>0.3</td>
<td>1.50</td>
</tr>
<tr>
<td>0.4</td>
<td>2.50</td>
</tr>
<tr>
<td>0.5</td>
<td>4.00</td>
</tr>
<tr>
<td>0.6</td>
<td>5.75</td>
</tr>
<tr>
<td>0.7</td>
<td>7.75</td>
</tr>
<tr>
<td>0.8</td>
<td>10.25</td>
</tr>
<tr>
<td>0.9</td>
<td>13.00</td>
</tr>
<tr>
<td>1.0</td>
<td>16.00</td>
</tr>
</tbody>
</table>

Source: practicalphysics.org

[Ignore data on p. 25.]
[It’s BAD data.]
Fitting Gravity Data

Example.  Modeling the dropping of a golf ball

Let’s use an experiment to test the gravity model from last time.

Use a camera to record the position every tenth of a second.

Data would be similar to the table →
It’s plotted in the scatterplot below.

<table>
<thead>
<tr>
<th>t</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.00</td>
</tr>
<tr>
<td>0.1</td>
<td>0.25</td>
</tr>
<tr>
<td>0.2</td>
<td>0.75</td>
</tr>
<tr>
<td>0.3</td>
<td>1.50</td>
</tr>
<tr>
<td>0.4</td>
<td>2.50</td>
</tr>
<tr>
<td>0.5</td>
<td>4.00</td>
</tr>
<tr>
<td>0.6</td>
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</tr>
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<td>16.00</td>
</tr>
</tbody>
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[Ignore data on p. 25.]
[It's BAD data.]
Fitting Gravity Data

These data seem to fit a (type of function).
Fitting Gravity Data

These data seem to fit a (type of function). How can we be sure?
Fitting Gravity Data

These data seem to fit a \( [\text{type of function}] \). How can we be sure?

1. Plot distance as a function of \( t^2 \).

![Graph of y as a function of t]

Position (y) | Time (t)
---|---
0.0 | 0.0
2.0 | 0.2
4.0 | 0.4
6.0 | 0.6
8.0 | 0.8
10.0 | 1.0

15.0
10.0
5.0
Fitting Gravity Data

These data seem to fit a \underline{type of function}. How can we be sure?

1. Plot distance as a function of $t^2$.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$t^2$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>0.1</td>
<td>0.25</td>
<td>0.25</td>
</tr>
<tr>
<td>0.2</td>
<td>0.75</td>
<td>0.75</td>
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<tr>
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<td>1.50</td>
<td>1.50</td>
</tr>
<tr>
<td>0.4</td>
<td>2.50</td>
<td>2.50</td>
</tr>
<tr>
<td>0.5</td>
<td>4.00</td>
<td>4.00</td>
</tr>
<tr>
<td>0.6</td>
<td>5.75</td>
<td>5.75</td>
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<td>10.25</td>
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</tr>
<tr>
<td>0.9</td>
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<td>13.00</td>
</tr>
<tr>
<td>1.0</td>
<td>16.00</td>
<td>16.00</td>
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Fitting Gravity Data

These data seem to fit a \underline{type of function}. How can we be sure?

1. Plot distance as a function of $t^2$.
   
   Next, estimate the constant of proportionality.

\[ y \text{ as a function of } t \]

\[
\begin{array}{c|c|c}
 t & t^2 & y \\
\hline
 0.0 & 0.00 & 0.00 \\
 0.1 & 0.25 & 0.25 \\
 0.2 & 0.75 & 0.75 \\
 0.3 & 1.50 & 1.50 \\
 0.4 & 2.50 & 2.50 \\
 0.5 & 4.00 & 4.00 \\
 0.6 & 5.75 & 5.75 \\
 0.7 & 7.75 & 7.75 \\
 0.8 & 10.25 & 10.25 \\
 0.9 & 13.00 & 13.00 \\
 1.0 & 16.00 & 16.00 \\
\end{array}
\]
These data seem to fit a \( \text{(type of function)} \). How can we be sure?

1. Plot distance as a function of \( t^2 \).

Next, estimate the constant of proportionality.

\[
\begin{array}{|c|c|c|}
\hline
 t & t^2 & y \\
\hline
 0.0 & 0.00 & \\
 0.1 & 0.25 & \\
 0.2 & 0.75 & \\
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 0.4 & 2.50 & \\
 0.5 & 4.00 & \\
 0.6 & 5.75 & \\
 0.7 & 7.75 & \\
 0.8 & 10.25 & \\
 0.9 & 13.00 & \\
 1.0 & 16.00 & \\
\hline
\end{array}
\]

This implies \( y \approx ____t^2 \).
Fitting Gravity Data

**Key Concept:** When fitting data to a function \( y = Ct^k \),

An alternate method is:

2. \( \star \) Plot the log of distance as a function of log of time. \( \star \)
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► WHY? Suppose $y = Ct^k$. Taking a logarithm of both sides,
$\ln y = \ln(Ct^k) =$
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**Conclusion:** To approximate $C$ and $k$,

► First, calculate $\ln y$ and $\ln t$
for each datapoint.
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\[ \ln y = \ln(Ct^k) = \]

**Conclusion:** To approximate \( C \) and \( k \),

► First, calculate \( \ln y \) and \( \ln t \) for each datapoint.
► Fit the transformed data to a line.

\[
\ln y \approx 2 \ln t + 2.8
\]
Fitting Gravity Data

Key Concept: When fitting data to a function \( y = Ct^k \),
An alternate method is:

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▷ WHY? Suppose \( y = Ct^k \). Taking a logarithm of both sides,
\[
\ln y = \ln(Ct^k) =
\]

Conclusion: To approximate \( C \) and \( k \),

▷ First, calculate \( \ln y \) and \( \ln t \)
for each datapoint.
▷ Fit the transformed data to a line.
   ▷ The slope is an approximation for \( k \).
   ▷ The \textit{y}-intercept approximates \( \ln C \).

\[
\ln y \approx 2 \ln t + 2.8
\]
Fitting Gravity Data

We have determined that our gravity model

\[ y(t) = 16t^2 \]

appears to model the dropping of a golf ball.
Fitting Gravity Data

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\[ y(t) = 16t^2 \]
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Example. Raindrops
Fitting Gravity Data

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**Example.** Raindrops—Our model gives their position as \( y(t) = 16t^2 \).
Fitting Gravity Data

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Example. Raindrops—Our model gives their position as \( y(t) = 16t^2 \).
A raindrop falling from 1024 feet would land after \( t = 8 \) seconds.
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Example. Raindrops—Our model gives their position as \( y(t) = 16t^2 \).
A raindrop falling from 1024 feet would land after \( t = 8 \) seconds.
However, an experiment shows that the fastest drop takes 40 seconds, and that drops fall at different rates depending on their size.
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Even if we have a good model for one situation doesn’t mean it will apply everywhere.
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A raindrop falling from 1024 feet would land after \( t = 8 \) seconds.
However, an experiment shows that the fastest drop takes 40 seconds,
and that drops fall at different rates depending on their size.

Even if we have a good model for one situation doesn’t mean it will
apply everywhere. **We always need to question our assumptions.**

—Extensive gravity discussion in Section 1.3.—
Modeling Population Growth

Example. Modeling the size of a population.

We would like to build a simple model to predict the size of a population in 10 years.
Modeling Population Growth

**Example.** Modeling the size of a population.

We would like to build a **simple** model to predict the size of a population in 10 years.

- A very **macro**-level question.
Example.  

Modeling the size of a population.

We would like to build a **simple** model to predict the size of a population in 10 years.

A very **macro**-level question.

**Definitions:** Let $t$ be time in years; $t = 0$ now.

$P(t) = $ size of population at time $t$.

$B(t) = $ number of births between times $t$ and $t + 1$.

$D(t) = $ number of deaths between times $t$ and $t + 1$.

Therefore, $P(t + 1) = $__________________________.

<table>
<thead>
<tr>
<th>Definitions imply</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(4) = $</td>
</tr>
<tr>
<td>$B\left(\frac{1}{2}\right) = $</td>
</tr>
<tr>
<td>$B(5) - D(5) = $</td>
</tr>
</tbody>
</table>
Modeling Population Growth

Example. Modeling the size of a population.
We would like to build a simple model to predict the size of a population in 10 years.

- A very macro-level question.

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Therefore, $P(t + 1) =$ ________________.

Assumption: The birth rate and death rate stay constant.

That is, the birth rate $b = \frac{B(t)}{P(t)}$ and death rate $d = \frac{D(t)}{P(t)}$ are constants.
Modeling Population Growth

Example. Modeling the size of a population.
We would like to build a simple model to predict the size of a population in 10 years.

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\( D(t) = \) number of deaths between times \( t \) and \( t + 1 \).

Therefore, \( P(t + 1) = \) ____________________________.

Assumption: The birth rate and death rate stay constant.

That is, the birth rate \( b = \frac{B(t)}{P(t)} \) and death rate \( d = \frac{D(t)}{P(t)} \) are constants.

Assumption: No migration.
Population Growth

Therefore,

\[ P(t + 1) = P(t) \left[ \frac{P(t)}{P(t)} + \frac{B(t)}{P(t)} - \frac{D(t)}{P(t)} \right]. \]

Under our assumptions,

\[ P(t + 1) = P(t)[1 + b - d]. \]
Population Growth

Therefore, 

\[ P(t + 1) = P(t) \left[ \frac{P(t)}{P(t)} + \frac{B(t)}{P(t)} - \frac{D(t)}{P(t)} \right]. \]

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\[ P(t + 1) = P(t)[1 + b - d]. \]

This implies: \( P(1) = \) ____________.
Population Growth

Therefore,

\[ P(t + 1) = P(t) \left[ \frac{P(t)}{P(t)} + \frac{B(t)}{P(t)} - \frac{D(t)}{P(t)} \right]. \]

Under our assumptions,

\[ P(t + 1) = P(t)[1 + b - d]. \]

This implies:

\begin{align*}
P(1) &= \_\_\_, \\
P(2) &= \_\_\_, \ldots
\end{align*}
Population Growth

Therefore, \[ P(t + 1) = P(t) \left[ \frac{P(t)}{P(t)} + \frac{B(t)}{P(t)} - \frac{D(t)}{P(t)} \right]. \]

Under our assumptions, \[ P(t + 1) = P(t)[1 + b - d]. \]

This implies: 
\[ P(1) = \text{___________}, \]
\[ P(2) = \text{___________}, \ldots \]
In general, \[ P(n) = \text{___________}. \]
Population Growth

Therefore, \[ P(t + 1) = P(t) \left[ \frac{P(t)}{P(t)} + \frac{B(t)}{P(t)} - \frac{D(t)}{P(t)} \right] \).

Under our assumptions, \[ P(t + 1) = P(t)[1 + b - d] \).

This implies: \[ P(1) = \] \underline{__________},

\[ P(2) = \] \underline{__________}, \ldots

In general, \[ P(n) = \] \underline{__________}.

Definition. The growth rate of a population is \( r = (1 + b - d) \). This constant is also called the Malthusian parameter.
Population Growth

Therefore, 
\[ P(t + 1) = P(t) \left( \frac{P(t)}{P(t)} + \frac{B(t)}{P(t)} - \frac{D(t)}{P(t)} \right). \]

Under our assumptions, 
\[ P(t + 1) = P(t)[1 + b - d]. \]

This implies: 
\[ P(1) = \underline{\hspace{2cm}}, \]
\[ P(2) = \underline{\hspace{2cm}}, \ldots \]

In general, 
\[ P(n) = \underline{\hspace{2cm}}. \]

Definition. The growth rate of a population is 
\[ r = (1 + b - d). \]

This constant is also called the Malthusian parameter.

A model for the size of a population is 
\[ P(t) = P(0)r^t, \]
where \( P(0) \) and \( r \) are constants.
Applying the Malthusian Model

Example 1. Suppose that the current US population is 317,420,000. Assume that the birth rate is 0.02 and the death rate is 0.01. What will the population be in 10 years?

Answer. Use $P(t) = P(0)r^t$: 
Applying the Malthusian Model

Example 1. Suppose that the current US population is 317,420,000. Assume that the birth rate is 0.02 and the death rate is 0.01. What will the population be in 10 years?

Answer. Use $P(t) = P(0)r^t$:

Refinement. Approx. US Growth Rate at http://www63.wolframalpha.com/input/?i=US+birth+rate

Resource: Wolfram Alpha, integrable directly into Mathematica.

Example 2. How long will it take the population to double?

Answer. Use $P(t) = P(0)r^t$:
Determining constants of exponential growth

**Goal:** Given population data, determine model constants.
Determining constants of exponential growth

**Goal:** Given population data, determine model constants.

- Take the logarithm of both sides of $P(t) = P(0)r^t$.
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Determining constants of exponential growth

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★ Important: Transformations distort distances between points, so verification of a fit should always take place on $y$ versus $x$ axes. ★
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The structure of the points in the residual plot give clues about whether the function fits the data well. Three common appearances:

1. **Random**: Residuals are randomly scattered at a consistent distance from axis. Indicates a good fit, as on previous page.

![Random Residual Plot](image)

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3. **Fanning**: Residuals small at first and get larger (or vice versa). Indicates non-constant variability (model better for small $x$?).
Suppose you have collected a set of known data points \((x_i, y_i)\), and you would like to estimate the \(y\)-value for an unknown \(x\)-value.

The name for such an estimation depends on the placement of the \(x\)-value relative to the known \(x\)-values.
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**Interpolation**

Inserting one or more \(x\)-values between known \(x\)-values.

**Extrapolation**

Inserting one or more \(x\)-values outside of the range of known \(x\)-values.
Interpolation vs. Extrapolation

The most common method for interpolation is taking a weighted average of the two nearest data points; suppose $x_1 < x < x_2$, then,

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- Confidence in extrapolated data is higher when closer to the range of known $x$-values.
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