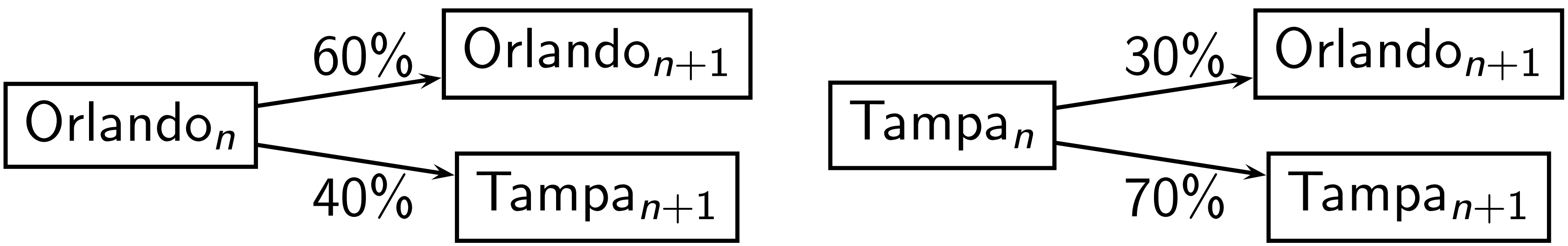
Markov Chains

A **Markov chain** is a sequence of random variables from some sample space, each corresponding to a successive time interval. From one time interval to the next, there is a *fixed* probability $a_{i,j}$ of transitioning from state j to state i. No transition depends on a past transition.

Keep track of these probabilities in an associated transition matrix A.

Example. Suppose you run a rental company based in Orlando and Tampa, Florida. People often drive between the cities; cars can be picked up and dropped off in either city. Suppose that historically,



What distribution of cars can the company expect in the long run?

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We will model this situation with a Markov Chain.

The historical data suggest that with a probability of 0.6, a car in Orlando at time n will be in Orlando at time $n+1$. Use this and the other expected transition probabilities to form the transition matrix A.

FROM:

Or Tm

TO: Or, Tm

$$\begin{bmatrix} & \\ & \end{bmatrix} = A,$$

- Let o_n be the probability that a car is in Orlando on day n
- Let t_n be the probability that a car is in Tampa on day n.

We can represent the distribution of cars at time n with the vector $\vec{\mathbf{x}}_n = \begin{bmatrix} o_n \\ t_n \end{bmatrix}$. And so, $\vec{\mathbf{x}}_{n+1} = \begin{bmatrix} o_{n+1} \\ t_{n+1} \end{bmatrix} = A \cdot \begin{bmatrix} o_n \\ t_n \end{bmatrix} = A\vec{\mathbf{x}}_n$.

Given an initial distribution $\vec{\mathbf{x}}_0 = \begin{bmatrix} o_0 \\ t_0 \end{bmatrix}$,

the expected distribution of cars at time n is $\vec{\mathbf{x}}_n =$ ____.

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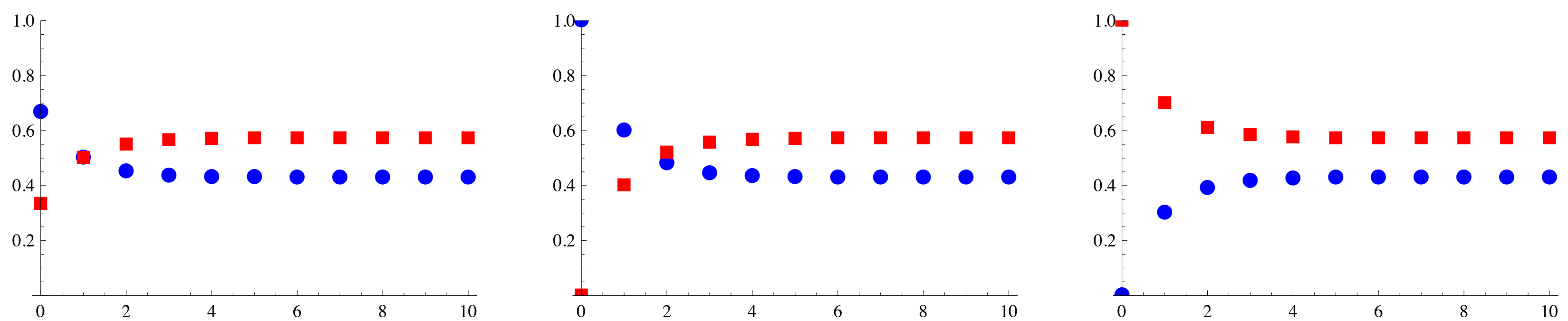
For example, if they company starts off with twice as many cars in Orlando as in Tampa, then $\vec{\mathbf{x}}_0 = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$, so we expect

$$\vec{\mathbf{x}}_1 = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}.$$

$$\vec{\mathbf{x}}_2 = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} \begin{bmatrix} \quad \\ \quad \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}.$$



How do we determine the expected distribution in the long run?

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Definition: Given a Markov Chain with transition matrix A, an **equilibrium distribution** is a vector $\vec{\mathbf{x}}_{eq}$ that satisfies $A\vec{\mathbf{x}}_{eq} = \vec{\mathbf{x}}_{eq}$.
[Linear Algebra: $\vec{\mathbf{x}}_{eq}$ *is an eigenvector corresponding to* $\lambda = 1$.]

In our example, the equilibrium distribution satisfies

$$\begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} \begin{bmatrix} o_{eq} \\ t_{eq} \end{bmatrix} = \begin{bmatrix} o_{eq} \\ t_{eq} \end{bmatrix}.$$

So solve: $0.6o_{eq} + 0.3t_{eq} = o_{eq}$ and $0.4o_{eq} + 0.7t_{eq} = t_{eq}$.
Both equations reduce to $0.3t_{eq} = 0.4o_{eq}$, so $o_{eq} = \frac{3}{4}t_{eq}$.

Conclusion: If the company has 7000 cars in all, they would expect that in the long run,____________________________________

In Markov Chains: ★ The sum of the entries in every column of A is 1, because the total probability of transitioning **from** state i is 1.

★ There is no general rule for what the row sum will be.