The language of optimization

Optimization questions cover a wide variety of situations.

Example. You are given the choice of **one** of the following candies.

Snickers bar	Gourmet chocolate square
Box of Mike & Ikes	Bounty (Coconut+Almond)
Swedish Fish	Tootsie roll lollypop
Kitkat Bar	Three Marshmallow Peeps
Licorice	Peanut M&M's

Fact: You face an optimization problem.

It has a **feasible set**: The set of all valid choices.

It has an **objective function**: The function we are optimizing over the feasible set.

$$f : \left\{ \begin{array}{c} \text{feasible} \\ \text{set} \end{array} \right\} \to \left\{ \begin{array}{c} \text{some measure} \\ \text{of goodness} \end{array} \right\}$$

Our feasible set is ________ and the objective function is ________.

Linear Optimization

We will focus on **linear optimization**.

- Feasible set: subset of $\mathbb{R}^n$ defined by **linear** inequalities.
- Objective function: **linear** combination of the input variables.

Comprehension goals:

- What is a linear program?
- Visualizing linear programs graphically.
- Understanding solutions graphically.
- Solving linear programs using *Mathematica*.
- Performing sensitivity analysis on linear programs.

Fertilizer example (p.253)

A fertilizer manufacturer uses nitrates and phosphates to make batches of two different kinds of fertilizer.

- Sod-King fertilizer needs 4 phosphates, 18 nitrates.
- Gro-Turf fertilizer needs 1 phosphate, 15 nitrates.

The company has 10 phosphates and 66 nitrates on hand.

The profit for one batch of Sod-King is \$1000.
The profit for one batch of Gro-Turf is \$500.

Question. How many batches of each should the company make to earn the most profit?

Initial thoughts?

- What do we need to know to make the problem precise?
- What intuition do you have for what the answer should be?

Fertilizer example (p.253)

Translate the problem into mathematics:
We must determine how many batches to make of each.

- Let x represent the number of batches of Sod-King made.
- Let y represent the number of batches of Gro-Turf made.

What are the constraints on what x and y can be?

- Phosphate constraint:
- Nitrate constraint:
- Non-negativity constraints:

What are we trying to maximize?

- Profit:

Linear Programs

$$\begin{array}{lrcl} \text{Maximize } 1000x + 500y & & & \\ \text{subject to} & 4x + y & \leq & 10 \\ \text{the constraints:} & 18x + 15y & \leq & 66 \\ & x & \geq & 0 \\ & y & \geq & 0 \end{array}$$

This is a **linear program**, an optimization problem of the form:

$$\begin{array}{lrcl} \multicolumn{4}{l}{\text{Maximize } c_1x_1 + c_2x_2 + \cdots + c_nx_n \text{ (the \textbf{objective function})}} \\ \text{subject to} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & \leq & b_1 \\ \text{(the \textbf{constraints}):} & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & \leq & b_2 \\ & \vdots & & \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & \leq & b_m \end{array}$$

Linear Programs

Maximize $c_1x_1 + c_2x_2 + \cdots + c_nx_n$ (the **objective function**)

subject to (the **constraints**):

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\leq b_2 \\ \vdots \quad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\leq b_m \end{aligned}$$

Notes about linear programs:

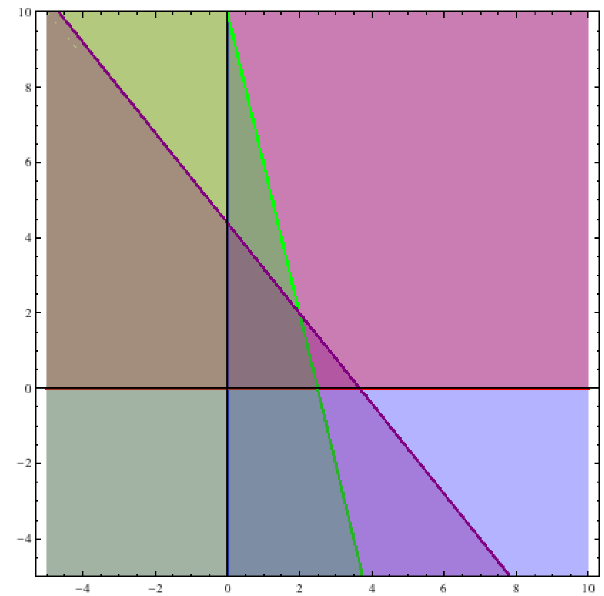
- Constraints may be of the form $\leq$, $=$, or $\geq$.
- The x_i variables are called **decision variables**.
- The decision variables can take on any real $\#$, not only integers.
- All constraints and the objective functions are *linear combinations* of the decision variables. (Coefficients are constants.)
- A linear program in the above form is "easy to solve".

Fertilizer example, graphically

Maximize $1000x + 500y$

subject to the constraints:

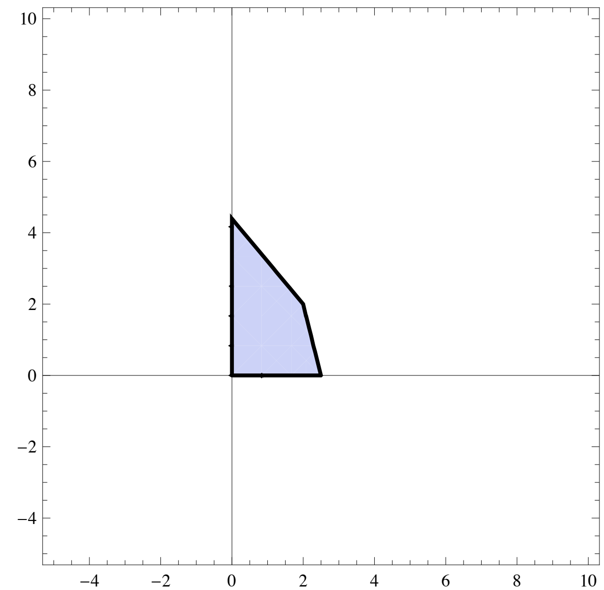
$$\begin{aligned} 4x + y &\leq 10 \\ 18x + 15y &\leq 66 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$



Let's consider our example graphically.

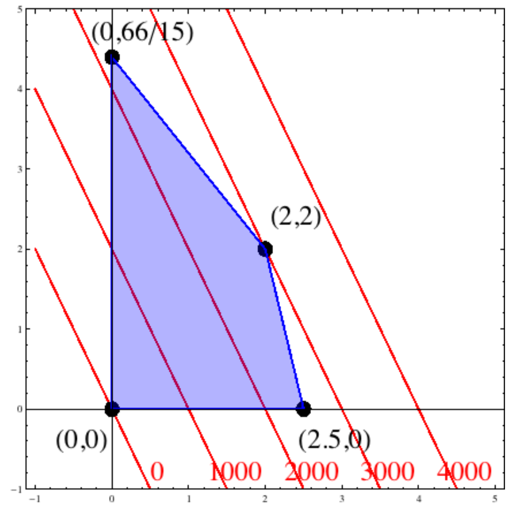
Definition: The set of points (x, y) that satisfy the constraints is called the **feasible region**.

- In general, points of form $(x_1, x_2, \ldots, x_n)$.
- Feasible region always a polytope. (Always has flat sides and is convex.)
- Feasible region may be bounded or unbounded; might be empty.



Fertilizer example, graphically

$$\begin{array}{ll} \text{Maximize } 1000x + 500y & \\ \text{subject to} & 4x + y \leq 10 \\ \text{the constraints:} & 18x + 15y \leq 66 \\ & x \geq 0 \\ & y \geq 0 \end{array}$$



⋆ The solution to the optimization problem will be the point in the feasible region that optimizes the objective function. ⋆

Is there a point in the feasible region such that $1000x + 500y = 2000$?
Is there a point in the feasible region such that $1000x + 500y = 4000$?

As we plot these lines of **constant objective**, we notice that

- They are parallel.
- If there is a feasible region, at least one line will intersect it.
- As we increase the "constant", the last place we touch the feasible region is

Linear Optimization

We have intuited the following theorem.

Theorem. The maximum (or minimum) in a linear program either:

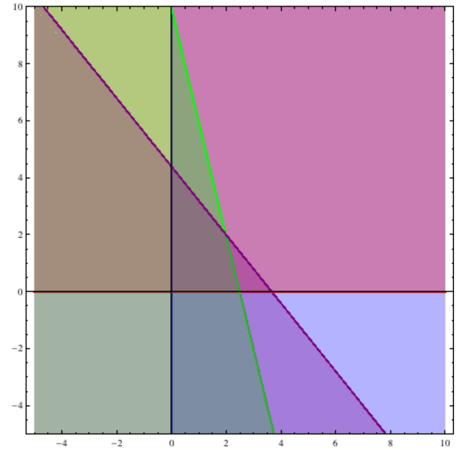
1. Occurs at a corner point of the feasible region.
2. Doesn't exist (then we call the problem unbounded)

Strategy for solving a linear optimization problem:

0. Determine the decision variables, objective function, and constraints.
1. Draw the feasible region.
2. Compute the coordinates of all corner points. (You must think!)
3. Evaluate the objective function at each corner point.
4. Pick out the optimum value.

Solution of fertilizer example

$$
\begin{aligned}
&\text{Maximize } p(x,y) = 1000x + 500y \\
&\text{subject to the constraints:} \quad
\begin{aligned}
4x + y &\le 10 \\
18x + 15y &\le 66 \\
x &\ge 0 \\
y &\ge 0
\end{aligned}
\end{aligned}
$$



1. Draw the feasible region. (Done!)
2. Compute the coordinates of all corner points.
   - Find the constraints that intersect; solve the associated equalities.
   - $x \ge 0$ and $y \ge 0$: $(0,0)$. **(Not all intersections!)**
   - $x \ge 0$ and $18x + 15y \le 66$: $(0, 22/5)$.
   - $y \ge 0$ and $4x + y \le 10$: $(5/2, 0)$
   - $18x + 15y \le 66$ and $4x + y \le 10$: $(2,2)$.
3. Evaluate the objective function at each corner point.
   - $p(0,0) = 0$
   - $p(0, 22/5) = 2200$
   - $p(5/2, 0) = 2500$
   - $p(2,2) = 3000$.
4. Pick out the optimum value. [Max value: \$3000, occurs at (2,2).]

Using *Mathematica* to solve a linear program

Once you have written your optimization problem as a linear program, you can use *Mathematica* to solve your problem.

Use either the `Maximize` or `Minimize` command.

Syntax: `Maximize[{`*obj*`,`*constr*`},`*vars*`]`

- *obj* is the objective function that you wish to optimize.
- *constr* are the set of all constraints, joined with `&&`'s (ANDs).
- *vars* is the set of variables.

```
In[1]:  Maximize[{1000 x + 500 y,
           x >= 0 && y >= 0 && 4 x + y <= 10 && 18 x + 15 y <= 66}, {x,y}]
Out[1]: {3000, {x -> 2, y -> 2}}
```

Output: Optimum value and optimum point.