Families of Graphs

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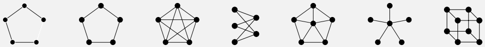

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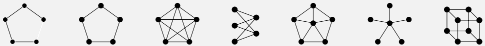

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We often try to find and/or count paths and cycles in a graph.

Question. What is the smallest path? Smallest cycle?

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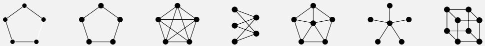

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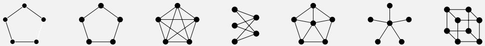

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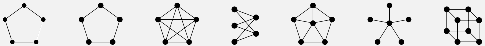

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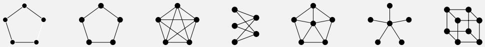
Families of Graphs



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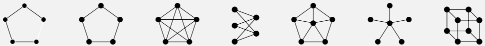

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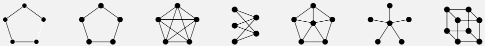

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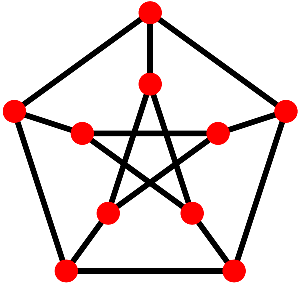
- **Cube graph $\square_n$**: The cube graph in n dimensions, $\square_n$, has 2^n vertices. We index the vertices by binary numbers of length n. Two vertices are adjacent when their binary numbers differ by exactly one digit.

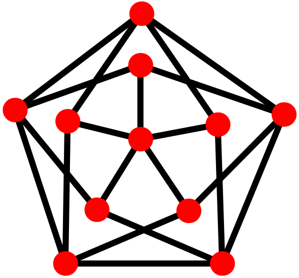
Special Graphs

Two graphs we will see on a consistant basis are:

Petersen graph P



Grötzsch graph Gr



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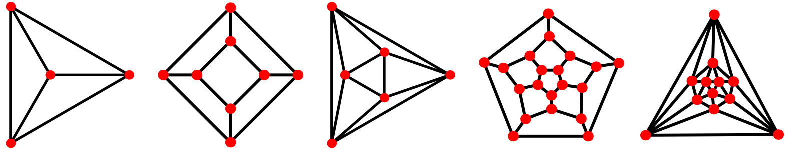
- The **Platonic graphs** are the Schlegel diagrams of the five platonic solids.



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Side note: The set of homomorphisms of a graph (isomorphisms into itself) is a measure of its symmetry. *Example.* 

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Consequence: Suppose $G = (V, E_1)$ and $G^c = (V, E_2)$. Then $E_1 \cap E_2 = \emptyset$ and $E_1 \cup E_2 = E(K_{|V|})$. (Recall K_n: complete graph.)

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Example. Show that the wheel W_6 contains a cycle of length 3, 4, 5, 6, and 7.

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Induced subgraphs of G are always subgraphs of G, but not vice versa.